

ON THE INTEGRABILITY ASPECTS OF THE SELF DUAL MEMBRANE

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December, 1996

ABSTRACT

The exact quantum integrability aspects of a sector of the membrane is investigated. It is found that spherical membranes moving in flat target spacetime backgrounds admit a class of integrable solutions linked to $SU(\infty)$ SDYM equations (dimensionally reduced to one temporal dimension). After a suitable ansatz, the SDYM equations can be recast in the form of the continuous Toda molecule equations whose symmetry algebra is the dimensional reduction of the $W_\infty \oplus \bar{W}_\infty$ algebra. The latter algebra is explicitly constructed. Highest weight representations are built directly from the infinite number of defining relations among the highest weight states of W_∞ algebras and the quantum states of the Toda molecule. Discrete states are also constructed. The full (dimensionally reduced) quantum $SU(\infty)$ YM theory remains to be explored.

PACS : 0465.+e; 02.40.+m

1. Introduction

The exact quantization program of the relativistic membrane has not been successful yet. The aim of the present work is a step forwards in that direction. A integrable sector of the membrane, the self dual membrane, can be quantized due to its equivalence to other known integrable models. The lightcone spherical supermembrane (in flat target spacetime backgrounds) is essentially equivalent to a $SU(\infty)$ Supersymmetric Gauge Quantum Mechanical Matrix Models (SGQMM's); i.e a $SU(\infty)$ super Yang Mills theory dimensionally reduced to one temporal dimension. For the particular case of Self Dual solutions (and other special class of solutions related to them) the bosonic sector of the SGQMM's can be mapped to a known integrable model : the $SU(\infty)$ continuous Toda molecule, that has the dimensionally reduced $W_\infty \oplus \bar{W}_\infty$ algebra as its symmetry.

Recently [1], exact solutions to $D = 11$ spherical (super) membranes moving in flat target spacetime backgrounds were constructed based on a particular class of reductions of Yang-Mills equations from higher dimensions to four dimensions [1,2]. The starting point was dimensionally-reduced Super Yang-Mills theories based on the infinite dimensional $SU(\infty)$ algebra. The latter algebra is isomorphic to the area-preserving diffeomorphisms of the sphere [3]. In this fashion the super Toda molecule equation was recovered preserving one supersymmetry out of the $N = 16$ expected. The expected critical target spacetime dimensions for the (super) membrane , $D = 27(11)$, was closely related to that of an anomaly-free non-critical (super) W_∞ string theory. A BRST analysis revealed that the spectrum of the membrane should have a relationship to the first unitary minimal model

of a W_N algebra adjoined to a critical W_N string in the $N \rightarrow \infty$ limit [1]. The class of particular solutions of the dimensionally-reduced $SU(\infty)$ YM equations studied are those of the type proposed by Ivanova and Popov [2] which bears a direct relationship to $SU(\infty)$ instanton solutions in $4D$ that permits a connection to the $SU(\infty)$ Toda molecule equation after an specific ansatz is made [1].

The Toda theory emerges in other contexts *beyond* the SDYM sector (the self dual membrane). It makes its appearance in noncritical W_∞ strings; i.e. W_∞ gravity. Conformal (affine) Toda theories are related through specific limits to conformal field theories and to their integrable off-critical perturbations [56]. The Toda theory is a very useful laboratory to explore what structures in $2+1$ dimensions correspond to integrable models in $1+1$ dimensions. The possibility that the membrane might be integrable has been put forward by Fairlie and Govaerts [44].

The quantum $4D$ membrane model has been studied by Jevicki [23] and he found that it is related to dilatonic-selfdual gravity plus matter. Self dual gravity can be obtained from $SU(\infty)$ SDYM by dimensional reduction [19, 24], and once more , self dual solutions play an important role. A review of $SU(\infty)$ SDYM and noncritical W_∞ strings [1] is described in the next section in connection to the Toda theory . In this section we review the contents of [1] and analyze in detail the role that noncritical W_∞ strings have in the theory of membranes. The critical dimensions, $D = 27, 11$ are recovered for the bosonic and supersymmetric case. Comments about the role that nonlinear noncritical W_∞ strings in the full theory are made at the end of **II**.

In **III** we discuss the classical solutions to the continuous Toda equation found by Saveliev [4] and make brief comments about the possibility of using the W_∞ co-adjoint orbit quantization method [25,26] to quantize the model exactly , after a Killing-symmetry reduction of quantum $4D$ self dual gravity. Then the continuum limit of the quantum A_{N-1} Toda theory [33] is performed where we make our only assumption and state that the asymptotic quantum states of the putative quantum continuous Toda molecule can be described solely in terms of the expectation values of the $\hat{\varphi}(t)$ operator which is present in the continuum and asymptotic limit of the results of [33].

In **IV** the explicit U_∞ algebra is constructed by direct dimensional reduction of the $W_\infty \oplus \bar{W}_\infty$ algebra. Defining relations that map the highest weights of the latter algebra into the quantum states of the Toda molecule are furnished in two representations.

In **V** a WKB-type semiclassical quantization is performed that yields exact solutions to the program outlined in **IV**. Discrete states are found. Finally in **VI** we present our conclusions.

2. Background

2.1 $SU(\infty)$ SDYM and the Toda Molecule

Based on the observation that the spherical membrane (excluding the zero modes) moving in D spacetime dimensions, in the light-cone gauge, is essentially equivalent to a $D - 1$ Yang-Mills theory, dimensionally reduced to one time dimension, of the $SU(\infty)$ group (see [8] for references); we look for solutions of the $D = 10$ Yang-Mills equations (dimensionally-reduced to one temporal dimension). For an early review on membranes see Duff [8] and the recent book by Ne'eman and Eizenberg [9].

Marquard et al [10] have shown that the light-cone gauge Lorentz algebra for the

bosonic membrane is anomaly free iff $D = 27$. The supermembrane critical dimension was found to be $D = 11$. To this date there is still some controversy about whether or not the (super) membrane is really anomaly free in these dimensions. They may suffer from other anomalies like $3D$ reparametrization invariance anomalies or global ones. What follows next does *not* depend on whether or not $D = 27, D = 11$ are truly the critical dimensions. What follows is just a straightforward quantization of a very special class of solutions of the dimensionally-reduced (to one temporal dimension) of $SU(\infty)$ YM equations, and which can be quantized exactly due to their equivalence to the exactly integrable quantum continuous Toda molecule, obtained as a dimensional-reduction of the original continuous $3D$ Toda theory [4,6] to $2D$ which is where W_∞ strings live; this clarifies how a $3D$ membrane can have a connection to a $2D$ W_∞ string.

We begin with the $D = 10$ YM equations dimensionally reduced to one dimension. Let us focus on the bosonic sector of the theory. The supersymmetric case can be also analyzed via solutions to the supersymmetric Toda theory which has been discussed in detail in the literature. The particular class of solutions one is interested in are those of the type analyzed by Ivanova and Popov. Given :

$$\partial_a F_{ab} + [A_a, F_{ab}] = 0. A_a^\alpha T_\alpha \rightarrow A_a(x^b; q, p). [A_a, A_b] \rightarrow \{A_a, A_b\}_{q,p}. \quad (2.1)$$

where the $SU(\infty)$ YM potentials [5] are obtained by replacing Lie-algebra valued potentials (matrices) by c number functions; Lie-algebra brackets by Poisson brackets w.r.t the two internal coordinates associated with the sphere; and the trace by an integral w.r.t these internal coordinates. In [1] we performed an ansatz following the results of Ivanova and Popov. The $a, b, \dots = 8$ are the transverse indices to the membrane after we performed the $10 = 2 + 8$ split of the original $D = 10$ YM equations.

After the dimensional reduction to one dimension is done we found that the following $D = 10$ YM potentials, \mathcal{A} , -which will be later expressed in terms of the $D = 4$ YM potentials, A_1, A_2, A_3 (A_0 can be gauged to zero) - are one class of solutions to the original $D = 10$ equations iff they admit the following relationship :

$$\mathcal{A}_1 = p_1 A_1. \mathcal{A}_5 = p_2 A_1. \mathcal{A}_3 = p_1 A_3. \mathcal{A}_7 = p_2 A_3. \quad (2.2a)$$

$$\mathcal{A}_2 = p_1 A_2. \mathcal{A}_6 = p_2 A_2. \mathcal{A}_0 = \mathcal{A}_4 = \mathcal{A}_8 = \mathcal{A}_9 = 0. \quad (2.2b)$$

where p_1, p_2 are constants and A_1, A_2, A_3 are functions of x_0, q, p only and obey the $SU(\infty)$ Nahm's equations :

$$\epsilon_{ijk} \frac{\partial A_k}{\partial x_0} + \{A_i, A_j\}_{q,p} = 0. i, j, k = 1, 2, 3. \quad (2.3)$$

Nahm's equations are also obtained directly from reductions of $D = 4$ Self Dual Yang-Mills equations to one dimension. The temporal variable $x_o = p_1 X_0 + p_2 X_4$ has a *correspondence*, not an identification, with the membrane's light-cone coordinate : $X^+ = X^0 + X^{10}$. We refer to Ivanova and Popov and to our results in [1,2] for details.

Expanding $A_y = \sum A_{yl}(x_o) Y_{l,+1}$. $A_{\bar{y}} = \sum A_{\bar{y}l}(x_o) Y_{l,-1}$; and A_3 in terms of Y_{l0} , the ansatz which allows to recast the $SU(\infty)$ Nahm's equations as a Toda molecule equation is [1] :

$$\{A_y, A_{\bar{y}}\} = -i \sum_{l=1}^{\infty} \exp[K_{ll'}\theta_{l'}] Y_{l0}(\sigma_1, \sigma_2). \quad A_3 = - \sum_{l=1}^{\infty} \frac{\partial \theta_l}{\partial \tau} \cdot Y_{l0}. \quad (2.4)$$

with $A_y = \frac{A_1 + iA_2}{\sqrt{2}}$. $A_{\bar{y}} = \frac{A_1 - iA_2}{\sqrt{2}}$.

Hence, Nahm's equations become :

$$-\frac{\partial^2 \theta_l}{\partial \tau^2} = e^{K_{ll'}\theta_{l'}}. \quad l, l' = 1, 2, 3, \dots \quad (2.5)$$

This is the $SU(N)$ Toda molecule equation in Minkowski form. The θ_l are the Toda fields where $SU(2)$ has been embedded minimally into $SU(N)$. $K_{ll'}$ is the Cartan matrix which in the continuum limit becomes : $\delta''(t - t')$ [4]. The solution of the Toda theory is well known to the experts by now. Solving for the $\theta_l(\tau)$ fields and plugging their values into the first term of eq-(2.4) yields an infinite number of equations -in the $N \rightarrow \infty$ limit- for the infinite number of "coefficients" $A_{yl}(x_o), A_{\bar{y}l}(x_o)$. This allows to solve for the YM potentials *exactly*. The ansatz [1] automatically yields the coefficients in the expansion of the A_3 component of the $SU(\infty)$ YM field given in the second term of (2.4). Upon quantization of the $SU(\infty)$ YM theory, the first term in eq-(2.4) is replaced by a commutator of two operators and as such the coefficients $A_{yl}(x_o), A_{\bar{y}l}(x_o)$ become operators as well. The Toda fields become also operators in the Heisenberg representation. We will discuss the quantization of the Toda theory via the W_∞ codajoint orbit method [25,26] below.

The continuum limit of (2.5) is

$$-\frac{\partial^2 \theta(\tau, t)}{\partial \tau^2} = \exp \left[\int dt' \delta''(t - t') \theta(\tau, t') \right]. \quad (2.6)$$

Or in alternative form :

$$-\frac{\partial^2 \Psi(\tau, t)}{\partial \tau^2} = \int \delta''(t - t') \exp[\Psi(\tau, t')] dt' = \frac{\partial^2 e^\Psi}{\partial t^2}. \quad (2.7)$$

if one sets $K_{ll'}\theta_{l'} = \Psi_l$. The last two equations are the dimensional reduction of the $3D \rightarrow 2D$ continuous Toda equation given by Saveliev :

$$\frac{\partial^2 u(\tau, t)}{\partial \tau^2} = -\frac{\partial^2 e^u}{\partial t^2}. \quad i\tau \equiv r = z + \bar{z}. \quad (2.8a)$$

Eq-(2.8a) is referred as the $SU(\infty)$ Toda *molecule* whereas

$$\frac{\partial^2 u(z, \bar{z}, t)}{\partial z \partial \bar{z}} = -\frac{\partial^2 e^u}{\partial t^2}. \quad (2.8b)$$

is the $3D$ continuous Toda equation which can be obtained as rotational Killing symmetry reductions of Plebanski equations for Self-Dual Gravity in $D = 4$. Eq-(2.8a) is an effective $2D$ equation and in this fashion the original $3D$ membrane can be related to a $2D$ theory (where the W_∞ string lives in) after the light-cone gauge is chosen.

The Lagrangian (and equations) for 4D SD gravity can be obtained from a dimensional reduction of the $SU(\infty)$ SDYM (an effective six-dimensional one) [19,24] :

$$\mathcal{L} = \int dz d\tilde{z} dy d\tilde{y} \frac{1}{2} (\Theta_{,y} \Theta_{,z} - \Theta_{,\tilde{y}} \Theta_{,\tilde{z}}) + \frac{1}{3} \Theta \{ \Theta_{,y}, \Theta_{,\tilde{y}} \}. \quad (2.9)$$

where $\Theta(z, \tilde{z}, y, \tilde{y})$ is Plebanski's second heavenly form and the Poisson brackets are taken w.r.t y, \tilde{y} variables. A real slice can be taken by setting : $\tilde{z} = \bar{z}, \tilde{y} = \bar{y}$.

A rotatinal Killing symmetry reduction, $t \equiv y\tilde{y}$, yields the Lagrangian for the 3D Toda theory and a futher dimensional reduction $z + \tilde{z} = r$ gives the Toda molecule Lagrangian.

From [19] we can find the explicit map between Θ and $\rho(r, t)$

$$A_y = \Theta_{,y}; \quad A_{\bar{y}} = \Theta_{,\bar{y}}; \quad A_z = -\Theta_{,\bar{y}} + f(y, \bar{y}, z); \quad A_{\bar{z}} = \Theta_{,y} + g(y, \bar{y}, \bar{z}). \quad (2.10)$$

where f, g are integration "constants". In the gauge $A_o = 0 \Rightarrow A_z = A_{\bar{z}}$ and the two functions, f, g are constrained to obey the latter condition plus the additional relations stemming from the original SDYM equations. Therefore, f, g are fully determined.

From our ansatz (2.4) one can read off the correspondence between the Plebanski Θ (after the corresponding reductions) and the Toda field $\rho(r, t)$:

$$\{ \Theta_{,y}, \Theta_{,\bar{y}} \} \rightarrow \frac{\partial^2}{\partial t^2} e^\rho; \quad \frac{\partial}{\partial r} (-\Theta_{,\bar{y}} + f(y, \bar{y})) \rightarrow \frac{\partial^2 \rho}{\partial r^2}; \quad \frac{\partial}{\partial r} A_z = \frac{\partial}{\partial r} A_{\bar{z}}. \quad (2.11a)$$

One could use the original Killing symmetry reduction of Plebanski first heavenly equation due to Boyer and Finley [57] and , also discussed by Park [58], that takes the original $r \equiv y\bar{y}; z, \bar{z}$ variables into the new ones $(t, w = \bar{z}, \bar{w} = z)$.; such that $r \equiv e^u(t, w, \bar{w})$ obeys the continuous Toda equation iff Ω obeys Plebanski first heavenly equation :

$$u_{,w\bar{w}} = e^u_{,tt}. \quad t \equiv r\Omega_{,r}. \quad (r\Omega_{,r})_{,r}\Omega_{,z\bar{z}} - r\Omega_{,rz}\Omega_{,r\bar{z}} = 1. \quad (2.11b)$$

A solution of the Toda equation, $u = u(t, w, \bar{w})$ upon inversion yields $t = r\Omega_{,r} = f(u, w, \bar{w})$ which defines implicitly Ω in terms of u through the function (upon inversion) $f(u, w, \bar{w})$. And, finally, one makes contact with Savaliev's Lagrangian of the Toda molecule :

$$\mathcal{L} = \int dt \frac{1}{2} \left(\frac{\partial^2 x}{\partial r \partial t} \right)^2 + e^{(\partial^2 x / \partial t^2)}. \quad \rho(r, t) \equiv \frac{\partial^2 x}{\partial t^2}. \quad (2.11c)$$

Eqs-(2.9-2.11) are the essential equations that allows to extract the exact quantization of the Toda theory via the W_∞ coadjoint orbit method described by [25]. Nissimov and Pacheva have shown that induced W_∞ gravity could be seen as a WZNW model. They derived the explicit form of the Wess-Zumino quantum effective action of chiral W_∞ matter coupled to a chiral W_∞ gravity background. The quantum effective action could be expressed as a geometric action on a coadjoint orbit of the deformed group of area-preserving diffs of the cylinder. A "hidden" $SL(\infty, R)$ Kac-Moody algebra was obtained as a consequence of the $SL(\infty, R)$ stationary subgroup of the W_∞ codajoint orbit. Yamagishi and Chapline earlier [26] proved that an induced 4 D self-dual quantum gravity could be obtained via the W_∞ coadjoint orbit method. An effective quantum action (constructed in

the twistor space) was explicitly obtained as an infinite sum of two-dimensional effective Lagrangians with Polyakov two-dim lightcone gauge gravity as its first term (having a hidden $SL(2, R)$ Kac-Moody symmetry). The higher order terms are the result of the central extensions of the W_∞ algebra.

The crucial advantage that the W_∞ coadjoint orbit method has in the quantization of the continuous Toda theory is that an Operator Quantization method is extremely difficult. The ordinary Liouville theory, an $SL(2, R)$ Toda theory, is a notoriously difficult example. Its quantization using operator methods has taken years. Very recently, Fujiwara, Igarashi and Takimoto [21] have shown using exact operator solutions for quantum Liouville theory, based on canonical free field methods, that the exact solutions proposed by Otto and Weight [22] are correct to all orders in the cosmological constant. They found that the hidden (quantum group) exchange algebra found by Gervais and Schnitger [20], $U_q(sl(2))$, was essential in order to maintain locality and the operator form of the field equation. In the continuous Toda case one expects a hidden quantum group $U_q(sl(\infty))$ structure. Not surprisingly, the appearance of the hidden $U_q(sl(\infty))$ must stem from the hidden $SL(\infty, R)$ Kac-Moody algebra associated with the stability subgroup of the W_∞ coadjoint orbit method.

The quantum effective action of the Toda theory is directly obtained from the exact results of [25,26] by the straightforward reduction given in eqs-(2.9-2.11) : one reads off the quantum effective action for the Toda theory directly from the “dictionary” between the Θ and the ρ established in (2.9-2.11) *after* performing the Darboux change of coordinates given by Plebanski. It is the latter that expresses the second heavenly form, Θ in terms of the first heavenly form, Ω . This is needed because the Chapline-Yamagishi quantum effective action is given in terms of Ω .

Having discussed the importance of the Toda theory we proceed to study the role that noncritical W_∞ strings have in the membrane quantization.

2.2 A Membrane Sector as a Non Critical W_∞ String

In [1] we established the correspondence between the target space-times of non-critical W_∞ strings and that of membranes in $D = 27$ dimensions. The supersymmetric case was also discussed and $D = 11$ was retrieved. We shall review in further detail the construction [1].

The relevance of developing a W_∞ conformal field theory (with its quantum group extensions) has been emerging over the past years [28]. It was shown in [11] that the effective induced action of W_N gravity in the conformal gauge takes the form of a Toda action for the scalar fields and the W_N currents take the familiar free field form. The same action can be obtained from a constrained $WZNW$ model (modulo the global aspects of the theory due to the topology. Tsutsui and Feher have shown that richer structures emerge in the reduction process [29]) by a quantum Drinfeld-Sokolov reduction process of the $SL(\infty, R)$ Kac-Moody algebra at the level k . Each of these quantum Toda actions possesses a W_N symmetry.

The authors [11] coupled W_N matter to W_N gravity in the conformal gauge, and integrating out the matter fields, they arrived at the induced effective action which was precisely the same as the Toda action. It is not surprising that the Toda theory is related to 4 D Self-Dual gravity. In what follows, by W_N string we mean the string associated to

WA_{N-1} algebra.

In general, non-critical W_N strings are constructed the same way : by coupling W_N matter to W_N gravity. The matter and Liouville sector (stemming from W_N gravity) of the W_N algebra can be realized in terms of $N-1$ scalars, ϕ_k, σ_k repectively. These realizations in general have background charges which are fixed by the Miura transformations [12,13]. The non-critical string is characterized by the central charges of the matter and Liouville sectors, c_m, c_L . To achieve a nilpotent BRST operator these central charges must satisfy :

$$c_m + c_L = -c_{gh} = 2 \sum_{s=2}^N (6s^2 - 6s + 1) = 2(N-1)(2N^2 + 2N + 1). \quad (2.12)$$

In the $N \rightarrow \infty$ limit a zeta function regularization yields $c_m + c_L = -2$.

The authors [13] have shown that the BRST operator can be written as a sum of nilpotent BRST operators , Q_N^n , and that a nested basis can be chosen either for the Liouville sector or the matter sector but not for both. If the nested basis is chosen for the Liouville sector then [13] found that the central charge for the Liouville sector is :

$$c_L = (N-1)[1 - 2x^2 N(N+1)]. \quad (2.13)$$

were x is an *arbitrary* parameter which makes it possible to avoid the relation with the W_N minimal models if one wishes to . Later we will show explicitly that the value of c_L coincides precisely with the value of the central charge of a quantum Toda theory obtained from a quantum-Drinfeld-Sokolov reduction of the $SL(\infty, R)$ Kac-Moody algebra at the level k such that $k + N = \text{constant}$ (a constant that can be computed exactly) in the $N \rightarrow \infty$ limit.

By choosing , if one wishes, x appropriately one can, of course, get the q^{th} unitary minimal models by fixing x^2 to be :

$$x_o^2 = -2 - \frac{1}{2q(q+1)}. \quad (2.14)$$

where q is an integer. In this case, since $c_m + c_L = -c_{gh}$, the central charge for the matter sector must be :

$$c_m = (N-1)\left(1 - \frac{N(N+1)}{q(q+1)}\right). \quad (2.15)$$

which corresponds precisely to the q^{th} minimal model of the W_N string as one intended to have by choosing the value of x_o^2 . In the present case one has the freedom of selecting the minimal model since the value of q is arbitrary. If $q = N$ then $c_m = 0$ and the theory effectively reduces to that of the “critical” W_N string. Conversely, if one chooses for the nested basis that corresponding to the matter sector instead of the Liouville sector, the roles of “matter” and “Liouville” are reversed. One would then have $c_L = 0$ instead.

Noncritical strings involve two copies of the W_N algebra. One for the matter sector and other for the Liouville sector. Since W_N is nonlinear, one cannot add naively two realizations of it and obtain a third realization. Nevertheless there is a way in which this

is possible [13]. This was achieved by using the nested sum of nilpotent BRST operators, Q_N^n . One requires to have all the matter fields, ϕ_k ; the scalars of the Liouville sector in the nested basis, $\sigma_{n-1}, \dots, \sigma_{N-1}$ plus the ghost and antighost fields of the spin $n, n+1, \dots, N$ symmetries where n ranges between 2 and N . Central charges were computed for each set of the nested set of stress energy tensors, T_N^n depending on all of the above fields which appear in the construction of the BRST charges : Q_N^n .

In order to find a spacetime interpretation, the coordinates X^μ , must be related to a very specific scalar field of the Liouville sector (since one decided to choose the nested basis in the Liouville sector) and that field is σ^1 . It is this central charge, associated with the scalar field σ^1 , that *always* appears through its energy momentum tensor in the Miura basis. Because σ_1 always appears through its energy momentum tensor, it can be replaced by an effective T_{eff} of any conformal field theory as long as it has the same value of the central charge, $c = 1 + 12\alpha^2 \equiv 1 - 12x^2$, where α is a background charge.

$$T(\sigma^1) = -\frac{1}{2}\left(\frac{\partial\sigma_1}{\partial z}\right)^2 - \alpha\frac{\partial^2\sigma_1}{\partial z^2}. \quad (2.16)$$

In particular, having D worldsheet scalars, X^μ , with a background charge vector, α_μ :

$$T_{eff} = -\frac{1}{2}\partial_z X^\mu \partial_z X_\mu - \alpha_\mu \partial_z^2 X^\mu. \quad c_{eff} = D + 12\alpha_\mu \alpha^\mu = 1 + 12\alpha^2. \quad (2.17)$$

For example, in the critical W_∞ string case, one is bound to the unitary minimal models [12,13] and one must pick for central charge associated with the scalar, σ_1 , the value $\alpha^2 = -x^2 = -x_o^2$ given by (2.14). The explicit value of c of the critical W_∞ string is obtained :

$$c_{crit} = 1 + 12(\alpha_o)^2 = 1 - 12x_o^2 = 1 - 12\left(-2 - \frac{1}{2q(q+1)}\right) = 25; q = N \rightarrow \infty. \quad (2.18)$$

In the case of the ordinary critical string, $W_N = W_2$, $q = N = 2$, one has :

$$x_o^2 = -2 - \frac{1}{2q(q+1)} \rightarrow -2 - \frac{1}{12} \Rightarrow c_{eff} = 1 - 12x_o^2 = 1 + 25 = 26 = c_{crit}. \quad (2.19)$$

Since the parameter x in the non-critical string case is an *arbitrary* parameter that is no longer bound to be equal to x_o , the effective central charge in the non-critical W_N string is now $1 - 12x^2$ in contradistinction to the critical W_N string case : $1 - 12x_o^2$. Therefore, if one wishes to make contact with $D = 27$ X^μ scalars instead of $D = 25$ one can choose x in such a way that it obeys $1 - 12x^2 = (1 - 12x_o^2) + c_{m_o}$ where c_{m_o} will turn out to be the central charge of the $q = N + 1$ unitary minimal model of the W_N algebra. Clearly, if one had chosen $q = N$ instead, from eq-(2.15), one gets that $c_{m_o} \rightarrow 0$ and, as expected, the critical W_N string is recovered : $x^2 \rightarrow x_o^2(q = N)$ given by (2.14). If, in addition, one does *not* wish to break the target space-time Lorentz invariance one *cannot* have background

charges for the D X^μ coordinates. Therefore, for the case that $q = N + 1 \Rightarrow c_{m_o} = \frac{2(N-1)}{N+1}$ (instead of zero) is obtained from eq-(2.15), and the effective central charge is now :

$$c_{eff} \equiv 1 - 12x^2 = (1 - 12x_o^2) + c_{m_o} = [26 - (1 - \frac{6}{(N+1)(N+2)})] + [2\frac{N-1}{N+2}] \quad (2.20)$$

then one concludes that $D = 25 + 2 = 27 = c_{eff}$ is recovered in the $N \rightarrow \infty$ limit. The reason why one wrote the last term of eq-(2.20) in such a peculiar way will be clarified shortly. In this way we have shown that the expected critical dimension for the bosonic membrane background , $D = 27$, has the same number of X^μ coordinates as that of a non-critical W_∞ string background if one adjoins the $q = N + 1$ unitary minimal model of the W_N algebra to that of a critical W_N string spectrum in the $N \rightarrow \infty$ limit. From eq-(2.20) one also learns that

$$D = 2 + 25 = 27 = 1 - 12x^2 \Rightarrow 2x^2 = -\frac{13}{3}. \quad (2.21)$$

which will be important to find the value of the central charge of the Toda theory, below.

There are, of course, many other ways in which one could recover $D = 27$ X^μ besides the way shown in eq-(2.20). The latter is a particular combination involving the critical W_∞ string with the $q = N + 1$ unitary minimal models. It is important to study the other possibilities. Whatever these may be, these do not preclude the role that non-critical W_∞ strings have in the theory. The physical membrane spectrum has to contain a sector that should be related to a critical W_∞ string adjoined to a $q = N + 1$ unitary minimal model of the W_N algebra in the $N \rightarrow \infty$ limit. The full spectrum, moduli space of membrane vacua, etc...is far more vast than the slice furnished in (2.20). Our main point is that W_∞ conformal field theory, with its Kac-Moody extension, W_∞ gravity,... should contain important clues to classify the spectrum and the moduli space of vacua, in the same way that ordinary conformal field theory did for the string. More precisely, we will argue that it is the non-linear extensions of the W_∞ algebra that must be involved if one wishes to relate to Jevicki's recent results [23].

The critical W_∞ string [12] is a generalization of the ordinary string in the sense that instead of gauging the two-dimensional Virasoro algebra one gauges the higher conformal spin algebra generalization ; the W_∞ algebra. The spectrum can be computed exactly and is equivalent to an infinite set of spectra of Virasoro strings with unusual central charges and intercepts [12]. As stated earlier , the critical W_N string (linked to the A_{N-1} algebra) has for central charge the value ($q = N$) :

$$c = 1 - 12x_o^2 = 26 - (1 - \frac{6}{q(q+1)}) = 26 - (1 - \frac{6}{N(N+1)}) = 25 \quad (2.22)$$

where one has rewritten 25 as $26 - 1$ to be able to make contact with the Virasoro unitary minimal models, as well, given by the last term of eq-(2.20). This explains why the last term of (2.20) was written in such a peculiar way. Unitarity is achieved if the conformal-spin two-sector intercept is :

$$\omega_2 = 1 - \frac{k^2 - 1}{4N(N+1)}. \quad 1 \leq k \leq N-1. \quad (2.23)$$

A particular example of the above results is that in the ordinary non-critical (W_2) string there are many ways to have $c = 26$. Choose for arbitrary value $x^2 = -2$ as opposed to non-arbitrary value of $x_o^2 = -2 - \frac{1}{12}$ required by the $q = 2$ Virasoro unitary minimal model. The central charge of the Liouville sector (the nested basis) given by eq-(2.13) reads for $N = 2$: $c_L = 1(1 + 4.2.3) = 25 = 1 - 12x^2 = c_{eff}$, in this particular case the $c_L = c_{eff}$ and the $c_m = 26 - c_L = 1$. And viceversa, choosing the matter sector to be in the nested basis, reverses the roles of matter and Liouville, and one has $c_m = 25$; $c_L = 1$ which is the standard result that the ordinary $D = 26$ critical string can be seen as a non-critical string in $D = 25$ if one adjoins the Liouville mode that plays the role of the extra dimension.

It is not surprising in this picture of non-critical W_∞ strings and quantization of W_∞ gravity, to understand why there is the ubiquitous presence of W symmetry constraints in non-critical strings in $D \leq 1$ and matrix models in relation to the theory of integrable hierarchies. Paraphrasing [11] : “the partition function of the (multi) matrix model, Z , which is related to the partition function of some low-dimensional string theory, or equivalently, two-dimensional gravity coupled to some $d \leq 1$ matter system, \mathcal{Z} , via the relation $\mathcal{Z} = \log Z$, is a *special* solution of such integrable hierarchy. Special in the sense that it satisfies an extra constraint known as the string equation. In fact, Z is itself subject to an *infinite* number of constraints which form a Virasoro or W algebra “. Furthermore, W gravity coupled to W matter is related to topological coset models [11]. Whithin our picture described here it is no surprise to see the appearance of W symmetries in non-critical strings. One of the major advantages of W conformal field theories is that allows the passage! of the string $c = 1$ barrier.

An important remark is in order : we have to emphasize that one should *not* confuse c_{eff} with c_m, c_L in the same way that one must not confuse x^2 with x_o^2 . The ordinary (W_2) string is a very *special* case insofar that $c_{eff} = c_m$ or c_L depending on our choice for the nested basis. The $D = 27$ X^μ spacetime interpretation of the theory is *hidden* in the stress energy tensor of the σ^1 field $T(\sigma_1) \rightarrow T(X^\mu)$ with $c_{eff} = c(D) = D = 27$. And, in addition to the 27 X^μ , one still has the infinite number of scalars ϕ_1, ϕ_2, \dots and the infinite number of remaining fields, $\sigma_2, \sigma_3, \dots$ in the Liouville sector. Clearly the situation is vastly more complex than the string.

From eqs-(2.12,2.13) one can infer that the value of the central charge of the matter sector, after a zeta function regularization, is $c_m = 2 + \frac{1}{24}$. The value of $c_L = -4 - \frac{1}{24}$. And $c_{eff} = 27$. The value of c_m after regularization corresponds to the central charge of the first unitary minimal model of the WA_{n-1} after n is analytically continued to a negative value of $n = -146 \Rightarrow c(n) = 2(n-1)/(n+2) = 2 + \frac{1}{24}$ [14]. The value of c_L does not correspond to a minimal model but nevertheless corresponds to a very special value of c where the WA_{n-1} algebra truncates to that of the W algebra associated with non-compact coset models [14] for specific values of the central charge :

$$WA_{n-1} \Rightarrow W(2, 3, 4, 5) \sim \frac{\hat{sl}(2, R)_n}{\hat{U}(1)}. \quad (2.24)$$

this occurs at the value of $c(n) = 2(1 - 2n)/(n - 2) = -4 - \frac{1}{24}$ for $n = 146$. This is another important clue that W_∞ conformal field theory, with its Kac-Moody algebra extensions, rational and irrational, should reveal to us important information of the membrane spectrum and its moduli space of vacua.

The study of non-critical W_∞ strings is very complicated in general. For example, $\mathcal{W}(2, 3, 4, 5)$ strings are prohibitively complicated. One just needs to look into the cohomology of ordinary critical $W_{2,s}$ strings to realize this [12]. Nevertheless there is a way in which one can circumvent this problem when one restricts to the self dual solutions of the membrane. The answer lies in the integrability property of the continuous Toda equation [4]. In the previous subsection we have shown how the exact quantization of the Toda theory is automatically obtained by a straightforward dimensional reduction of the co-adjoint orbit quantization method described by [25,26]. Furthermore, the quasi-finite highest weight irreducible representations of the $W_{1+\infty}, W_\infty$ algebras [30] allows to classify the co-adjoint orbits associated with these representations.

We are going to proceed and calculate the value of the central charge of the Toda theory *without* the need to quantize it explicitly! The quantum Toda theory has for central charge the value given in (2.13) for the specific value of x^2 found in eq-(2.21). i.e. if the BRST quantization of the continuous Toda action is devoid of W_∞ anomalies the net central charge of the matter plus Toda sector must equal -2 as we saw in eq-(2.12); i.e. $c_L = c_{Toda}$ (after regularization).

The central charge of the quantum A_{N-1} Toda theory obtained from the quantum Drinfeld-Sokolov reduction of the (noncompact version of $SU(\infty)$) $SL(N, R)$ Kac-Moody algebra at level k [11] is :

$$c_{Toda} = (N - 1)(1 - N(N + 1)\frac{(k + N - 1)^2}{k + N}). \quad c_m + c_{toda} = -c_{gh} = c^{crit}. \quad (2.25)$$

Another way of rewriting eq-(2.25) is from the Drinfeld-Sokolov reduction process :

$$c_{DS} = (N - 1) - 12|\beta\rho - \frac{1}{\beta}\rho^\nu|^2; \quad \beta = \frac{1}{\sqrt{k + N}}. \quad (2.26)$$

ρ, ρ^ν are the Weyl weight vectors of the A_{N-1} Lie algebra and its dual, respectively. One can read now the value of x^2 directly from eq-(2.13) and eq-(2.25) by equating $c_{Toda} = c_L$:

$$2x^2 = -\frac{13}{3} = (\beta - \frac{1}{\beta})^2 = [\frac{1}{\sqrt{k + N}} - \sqrt{k + N}]^2 = \frac{(k + N - 1)^2}{(k + N)}. \quad (2.27)$$

The last equation allows us to compute explicitly the value of the coupling constant appearing in the exponential function that gives the interaction potential of the quantum Toda theory [6,15]. The Toda theory is conformally invariant and the conformally-improved stress energy tensor obeys a Virasoro algebra with an adjustable central charge whose value depends on the coupling constant of the exponential potential term appearing in the Toda action: $c(\beta)$. As pointed out in [11], it turns out that simply replacing the

BRST operators by a normal-ordered version does not yield a nilpotent operator. In addition one has to allow for possible (multiplicative) renormalizations of the stress-energy tensor appearing in the BRST charge. This is the origin of the $\beta = 1/\sqrt{k+N}$ factors.

One may immediately notice that the expression for (2.27) is invariant under the exchange of $\beta \rightarrow 1/\beta$, the exchange of strong/weak coupling, does not alter the value of the central charge. This is a good sign consistent with S duality symmetry of the alleged fundamental description of the membrane/string : M, F, \dots theory.

One can now relate the value of the level k , of the $SL(N, R)$ Kac-Moody algebra and N in such a way that $k + N = \beta^{-2}$ is a finite number when $N \rightarrow \infty$, :

$$2x^2 = (-13/3) = (\frac{1}{\sqrt{k+N}} - \sqrt{k+N})^2 = (\beta - \frac{1}{\beta})^2 \Rightarrow \beta^2 = \frac{-7 \pm \sqrt{13}}{6}. \quad (2.28)$$

The fact that $\beta = (k + N)^{-1/2}$ is purely imaginary should not concern us. There exist integrable field theories known as Affine Toda theories whose coupling is imaginary but possesses soliton solutions with real energy and momentum [15]. A natural choice is : $k = -\infty$ so that $k + N = \beta^{-2}$ is *finite* when $N \rightarrow \infty$.

The connection to the unitary Virasoro minimal models was established in eq-(2.22)(set $q = N + 1$) :

$$D - 2 = 25 = c_{string} - [1 - \frac{6}{q(q+1)}] = 26 - [1 - \frac{6}{(N+1)(N+2)}]. \quad (2.29)$$

This shall guide us in repeating the arguments for the supersymmetric case. Similar arguments leads to $D = 11$ in the supermembrane case [1]. The argument proceeds as follows :

Since 10 is the critical dimension of the ordinary superstring the value of the central charge when one has 10 worldsheet scalars and fermions is $10(1 + 1/2) = \frac{30}{2}$. In order to have the central charge of a critical super W_∞ string one requires to have also the central charge of the super Virasoro unitary minimal superconformal models : $c_{Virasoro} = 3/2$. The supersymmetric analog of the r.h.s of (2.29) is then :

$$10(1 + 1/2) - c_{superconformal} = \frac{30}{2} - \frac{3}{2} = \frac{27}{2}. \quad (2.30)$$

The supersymmetric analog of the term $c_{m_o} = \frac{2(N-1)}{N+1} \rightarrow 2$, is : $2(1 + 1/2) = 3$. One chooses the parameter x^2 in order to make contact with the bosonic sector of the $q = N + 1$ unitary minimal model of the super W_N algebra in the $N \rightarrow \infty$ limit. Writing down the corresponding supersymmetric analog of each single one of the terms appearing in the r.h.s of eq-(2.20), and the same for the l.h.s , one has that $D X^\mu$ and $D \psi^\mu$ (anticommuting spacetime vectors and world sheet spinors) *without* background charges yield a central charge $D(1 + 1/2) = \frac{3D}{2}$; Therefore, the supersymmetric extension of the corresponding terms of eq-(2.20) yields :

$$\frac{3D}{2} = [10(1 + 1/2) - 3/2] + [2(1 + 1/2)] = 33/2 \Rightarrow D = 11. \quad (2.31)$$

Concluding , one obtains the expected critical dimensions for the (super) membrane if one *adjoins* a $q = N + 1$ unitary (super) conformal minimal model of the (super) W_N algebra to a critical (super) W_N string spectrum in the $N \rightarrow \infty$ limit. This all suggests that a sector of the physical (super) membrane spectrum could be obtained exactly the same way. Hence, in a heuristic manner, we conjecture that : there is a sector of the physical (super) membrane spectrum that should be related to the non-critical (super) W_∞ string constructed above. Furthermore, the quantum (super) membrane must be related to the quantization of (super) W_∞ gravity. Further arguments that support our conjecture are given below.

What is required now is to quantize , upfront, the membrane and to formulate the no-ghost theorem in order to confirm, if true, our conjecture. This is a very difficult problem. The full-fledge membrane quantization is a more arduous task. As explained in the previous subsection, the self dual sector is just the $SU(\infty)$ Self Dual Yang-Mills theory that can be related to the Toda theory after the dimensional reduction. In view of our findings about interpreting the membrane as a non-critical W_∞ string with $c_{matter} + c_L = -2$ (eq-(2.12)) and $c_L = c_{Toda}$, (eqs-(2.13,2.25)), suggests that a large sector of the *physical* membrane spectrum, in addition to the one obtained by adjoining $q = N + 1 \rightarrow \infty$ unitary minimal W_N conformal matter to a critical W_∞ string, might be obtained by adjoining to the full quantized Toda (or SDYM theory) theory the remaining infinity of W_∞ conformal scalar matter fields : $\phi_1, \phi_2, \dots, \phi_k, \dots$ with the provision that the parameter x is fixed by $c_{eff}(x) = 1 - 12x^2 = 27$ and $c_m + c_L = -2$. (Similar considerations apply to the supermembrane).

Choosing a different value for x and integer values for c_{eff} suggests non-critical membrane backgrounds. An important difference between the ordinary critical string and the critical (super) membrane is that the latter requires an infinite number of fields in the Toda sector (a Liouville sector, c_L), an infinite number of fields in the matter sector (with central charge c_m) and the extra ($D = 11$) $D = 27$ spacetime coordinates (c_{eff}); in contradistinction to the critical string that only requires matter, $c_m = c_{eff}$ and no Liouville sector. This explains why $4D$ Self Dual Gravity (which upon dimensional reduction yields the continuous Toda) must be a crucial player in the membrane quantization as pointed out by Jevicki in a different context [23].

Some time ago we were able to show that the $D = 4$ $SU(\infty)$ (super) SDYM equations (an effective 6 dimensional theory) can be reduced to $4D$ (super) Plebanski's Self-Dual Gravitational equations with spacetime signatures $(4, 0); (2, 2)$. The symmetry algebra of $D = 4$ $SU(\infty)$ SDYM is a Kac-Moody extension of W_∞ as shown recently by [31]. In particular, new hidden symmetries were found which are affine extensions of the Lorentz rotations. These new symmetries form a Kac-Moody-Virasoro type of algebra. By rotational Killing-symmetry reductions one obtains the w_∞ algebra of the continuous Toda theory. For metrics with translational Killing symmetries one obtains the symmetry of the Gibbons-Hawking equations.

The relevance of Kac-Moody extensions of W_∞ algebras has also been pointed out by Jevicki who has shown that the $4D$ membrane in the lightcone gauge yields a four

dimensional world volume structure related to dilatonic-self dual gravity plus matter. The quantum theory is defined in terms of a $SU(\infty)$ Kac-Moody algebra. The quantization of $4D$ self dual gravity via the coadjoint orbit method has a hidden $SL(\infty, R)$ Kac-Moody algebra in the lightcone gauge. This is just the noncompact version of the $SU(\infty)$ Kac-Moody algebra. The presence of matter is also consistent with the presence of the matter fields ϕ_1, \dots, ϕ_k with a central charge c_m . The matter terms [23] appear as a current-current interaction, J^2 , where the four-dimensional field $J(x, t, \sigma^1, \sigma^2)$ found by Jevicki, originated from a $SU(\infty)$ current algebra that accounts for the extra two-indices σ^1, σ^2 .

This is similar to what we just found : $4D$ SD Gravity is a *reduction* of $D = 4$ $SU(\infty)$ $SDYM$ and the former is reduced to \rightarrow the continuous $3D$ Toda theory. The Toda sector appears in noncritical W_∞ strings with the infinite number of scalars (W_∞ conformal matter), ϕ_1, ϕ_2, \dots . This construction must be related to the membrane's spectrum . What is left is the presence of the scalar dilaton $\alpha(x, t)$ [23]. $2D$ dilatonic supergravity was studied by Ikeda within the context of a *nonlinear* gauge theory principle : one does not have a Lie bracket structure. The nonlinear gauge principle allowed the author to construct a non-linear bracket that led to *nonlinear* W_∞ algebras directly from nonlinear integrable perturbations of $4D$ self dual gravity. Hence, it seems that non-linear (but still integrable) perturbations of self dual gravity span a richer sector than W_∞ strings so it is very plausible that it is nonlinear noncritical W_∞ strings that bear a closer connection to the full membrane. What is warranted is to establish, if possible, the relationship between the matter sector of Jevicki's Hamiltonian and ours. Integrable but linear deformations of self dual gravity were studied by Strachan [32]. The original W_∞ algebra was constructed by [16]. The interpretation of such algebras as Moyal bracket deformations of the Poisson structure associated with the area-preserving diffs was found by [17,18].

This completes the review of [1]. We hope that we've clarified the interplay between $3D$ and $2D$ that appears after the light-cone gauge for the membrane is chosen and the importance that noncritical linear (nonlinear) W_∞ strings ; i.e. W_∞ conformal field theory must have in the understanding of the membrane. Quantum Group extensions will come as no surprise. W_∞ symmetries in string theory were also discussed by Zaikov [7].

3

Having reviewed the essential results of [1] compels us to look for classical solutions to the continuous Toda molecule equation and to implement the Quantization program presented in [33] for the finite nonperiodic Toda chain associated with the Lie Algebra A_{N-1} in the $N \rightarrow \infty$ limit. At the end of this section we will discuss further details about the role of the co-adjoint orbit quantization method in the quantization program of the continuous Toda theory.

The general classical solution to (2.8a) depending on two variables , say $r \equiv z_+ + z_-$ and t (not to be confused with time) was given by Saveliev [4]. The solution is determined by two arbitrary functions , $\varphi(t)$ and $d(t)$. It is :

$$\exp[-x(r, t)] = \exp[-x_o(r, t)] \left\{ 1 + \sum_{n>1} (-1)^n \sum_{\omega} \int \int \dots \exp[r \sum_{m=1}^n \varphi(t_m)] \prod_{m=1}^{m=n} dt_m d(t_m) \right\}$$

$$\left[\sum_{p=m}^n \varphi(t_p) \right]^{-1} \left[\sum_{q=m}^n \varphi(t_{\omega(q)}) \right]^{-1} \cdot [\epsilon_m(\omega) \delta(t - t_m) - \sum_{l=1}^{m-1} \delta''(t_l - t_m) \theta[\omega^{-1}(m) - \omega^{-1}(l)]] \}. \quad (3.1)$$

with : $\rho_o = \partial^2 x_o / \partial t^2 = r\varphi(t) + \ln d(t)$. This defines the boundary values of the solution $x(r, t)$ in the asymptotic region $r \rightarrow \infty$. θ is the Heaviside step-function. ω is any permutation of the indices from $[2, \dots, n] \rightarrow [j_2, \dots, j_n]$. $\omega(1) \equiv 1$. $\epsilon_m(\omega)$ is a numerical coefficient. See [4] for details. An expansion of (3.1) yields :

$$\exp[-x] = \exp[-x_o] \left\{ 1 - \mu + \frac{1}{2} \mu^2 + \dots \right\}. \quad (3.2)$$

where :

$$\mu \equiv \frac{d(t) \exp[r\varphi(t)]}{\varphi^2}. \quad (3.3)$$

A reasonable quantization proposal of the Quantum A_∞ (continuous) Toda chain could be obtained by taking the continuum limit of the general solution to the finite nonperiodic Toda chain associated with the Lie algebra A_N in the $N \rightarrow \infty$ limit. The real difficulty will be in proving that the continuum quantum limit of ρ obeys the operator quantum equations of motion as well. The quantization of sl_n Toda field theories in a periodic lattice was performed by Bonora and Bonservizi [56] and the quantum exchange algebra was studied. Babelon and Bonora [52] have also studied the Quantum Toda theory in a lattice version of the Coulomb gas picture and derived the exchange algebra in the Bloch basis.

We are *not* claiming that the naive continuum limit of the results of [33] will render the continuous quantum Toda theory in one full step. All what we need for our purposes is the assumption the asymptotic quantum states, $|\rho(in/out)\rangle$ solely depend on the function $\varphi(t)$. Upon quantization the latter becomes an operator $\hat{\varphi}(t)$ whose “eigenvalues” are ordinary functions $\varphi(t)$. The classical energy [4] given by eq-(3.9) solely depends on an arbitrary function of t : $\varphi(t)$. Upon quantization, the discrete energy states (assuming these exist and are physical) will restrict the allowed range of values for the latter function, and a sole class of possible values for $\varphi(t)$ is possible. In the standard Hydrogen atom, the discrete energy states are labeled in terms of the principal quantum number, n , and by the angular momentum, l and the quantum number m_l . The energy is a function of n .

Therefore, our *only* assumption is that $\hat{\varphi}(t)$ should be sufficient to parametrize the asymptotic quantum states of interest $|\rho(in/out)\rangle$ and that the energy, as well as other integrals of motion, can also be expressed in terms of the expectation values of the $\hat{\varphi}(t)$ operator. A special *subclass* of asymptotic states will be studied thoroughly in the next section in terms of quasi-finite highest weight irreducible representations (irreps) of W_∞ algebra [30,34,35].

We shall now take the continuum limit of eqs-(30-34) and eqs-(82-86) of [33] in order to get a grasp of the relevance of the $\hat{\varphi}(t)$ operator in connection with the asymptotic states.

$$\varphi_i \rightarrow x_o(r, t). \quad \psi_{j_s} \rightarrow \partial^2 x_o / \partial t_s^2 = r\varphi(t_s) + \ln d(t_s). \quad (3.4)$$

In the $r \rightarrow \infty$ limit the latter tends to $r\varphi(t_s)$. The continuum limit of (86) in [33] is :

$$\sum_{j_1 j_2 \dots j_n} \rightarrow \int \int \dots dt_1 dt_2 \dots dt_n. \mathcal{P}^1 \rightarrow [\sum \varphi(t_p) + O(\hbar)]^{-1} \cdot \mathcal{P}^2 \rightarrow [\sum \varphi(t_{\omega(q)}) + O(\hbar)]^{-1}. \quad (3.5)$$

Therefore, one just has to write down the quantum corrections to the two factors $[\sum \varphi]^{-1}$ appearing in eq-(3.1) above. One must replace the first factor by a summation from $p = m$ to $p = n$ of terms like :

$$[\varphi(t_p) + O(\hbar)] \rightarrow \varphi(t_p) - \frac{i\hbar}{w(t_p)} \left[\frac{1}{w(t_p)} \right]_{,t_p t_p} - i\hbar \sum_{l=p+1}^n \frac{1}{w(t_l)} \left[\frac{1}{w(t_l)} \right]_{,t_l t_l}. \quad (3.6)$$

and the second factor by a summation from $q = m$ to $q = n$ of terms like :

$$[\varphi(t_{\omega(q)}) + O(\hbar)] \rightarrow [eq (3.6) : p \rightarrow \omega(q)] + i\hbar \delta(t - t_{\omega(q)}) - i\hbar \sum_{l=1}^{q-1} [w(t_{\omega(l)})]^{-1} \left[\frac{1}{w(t_{\omega(l)})} \right]_{,t_{\omega(l)} t_{\omega(l)}} + i\hbar \sum_{l=q+1}^n [w(t_{\omega(l)})]^{-1} \left[\frac{1}{w(t_{\omega(l)})} \right]_{,t_{\omega(l)} t_{\omega(l)}} \quad (3.7)$$

where $w(t)$ is a positive function that is the continuum limit of eqs-(30,34) of [33]. What one has done is to replace :

$$\hat{k}_{j_m j_l} \equiv \frac{\tilde{k}_{j_m j_l}}{w_{j_l} w_{j_m}} \rightarrow \int dt_m \frac{\delta''(t_m - t_l)}{w(t_l) w(t_m)} = \frac{1}{w(t_l)} \left[\frac{1}{w(t_l)} \right]_{,t_l t_l} \quad (3.8a)$$

in all the equations in the continuum limit. One may smear out the delta functions which appears in eq-(3.7) if one wishes so that the denominators of eq- (3.1) are well defined.

These are the naive quantum corrections to the classical solution $\rho = \partial^2 x / \partial t^2$ where $x(r, t)$ is given in (3.1). These are the continuum limits of eqs-(82-86) of [33]. It is important to realize that one must not add quantum corrections to the $\varphi, d(t)$ appearing in the terms $\exp[r \sum \varphi]$ and x_o of (3.1). The former are two arbitrary functions which parametrize the space of classical solutions. Upon quantization it follows from eq-(31) of [33] that $\varphi(t), d(t)$ become r independent ("time" independent) operators obeying the equal r (time) commutation relations given by eq-(33) of [33] in the continuum limit :

$$[\hat{\varphi}(t), l n \hat{d}(t)] = -i\hbar \frac{1}{w(t)} \left[\frac{1}{w(t)} \right]_{,tt}. \quad (3.8b)$$

Therefore, ρ and x acquire \hbar quantum corrections given by (3.6,3.7) through the c -number function $w(t)$ terms and depend on the non-commuting operators given by $\hat{\varphi}(t), l n \hat{d}(t)$. Hence, ρ or x should be seen as quantum operators acting on the Hilbert space of states associated with the quantization of the continuous Toda field : $\rho(r, t)$. Such states are always labeled as $|\rho >_{\varphi(t), d(t)}$. For convenience purposes we shall omit the suffix from now on but we should keep it in mind. Upon quantization, \hbar appears and associated with Planck's constant a new parametric function has to appear : $w(t)$. One has to incorporate

also the coupling constant β in all of the equations . This is achieved by rescaling the continuous Cartan matrix by a factor of β so that $\partial^2 x / \partial t^2$ and $\partial^2 x_o / \partial t^2$ are rescaled by a factor of β ; i.e. $r\varphi$ acquires a factor of β and $d(t) \rightarrow d(t)^\beta$. Since β is pure imaginary, for convergence purposes in the $r = \infty$ region we must have that $\beta\varphi < 0 \Rightarrow \varphi = i\varphi$ also. In the rest of this section we will work without the β factors and only reinsert them at the end of the calculations. There is nothing unphysical about this value of β as we said earlier.

One of the integrals of motion is the energy. The continuous Toda chain is an exact integrable system in the sense that it possesses an infinite number of functionally independent integrals of motion : $I_n(p, \rho)$ in involution. i.e. The Poisson brackets amongst I_n, I_m is zero. Since these are integrals of motion, they do not depend on r . These integrals can be evaluated most easily in the asymptotic region $r \rightarrow \infty$. This was performed in [4] by Saveliev for the case that $\varphi(t)$ was a negative real valued function which simplified the calculations. For this reason the energy eigenvalue given in [4] must now be rescaled by a factor of β^2 :

$$E = \beta^2 \int_0^{2\pi} dt \left(\int^t dt' \varphi(t') \right)^2. \quad (3.9)$$

where we have chosen the range of the t integration to be $[0, 2\pi]$. Since $\beta\varphi < 0 \Rightarrow \beta^2\varphi^2 > 0$ and the energy is positive. We insist, once more, that t is a parameter which is not the physical time and that $\varphi(t)$ does not acquire quantum corrections. The latter integral (3.9) is the eigenvalue of the Hamiltonian which is one of the Casimir operators for the irreducible representations of A_N in the $N \rightarrow \infty$ limit.

The authors [47] found discrete energy levels for the quantum mechanical $SU(N)$ YM model. They did emphasize that discontinuities can occur in the $N = \infty$ limit. The reason that for finite N a discrete spectrum occurs is due to the existence of a non-zero value of the Casimir energy. The classical membrane admits continuous deformations of zero area with no energy expense . Quantum mechanically, this classical instability is cured by quantum effects and any wave-function gets stuck in the potential valleys that become increasingly narrow the farther out one gets. Finite-energy wave functions fall-off rapidly and the quantum Hamiltonian is purely discrete. The ground state energy corresponds to the finite non-zero Casimir energy : the point beyond which no further deformations of the membrane into long stringlike configurations is possible. In the supermembrane case the Casimir energy is zero and a continuous spectrum emerges. However, this was performed at a *finite* value of N . Here we are discussing the different case ; what happens at the $N = \infty$ limit.

As stated in **II**, the exact quantization of the continuous Toda Field is also possible via the W_∞ Coadjoint Orbit Method of 4D Self Dual gravity [25,26]. The lightcone-gauge effective 2D gravitational action derived by Polyakov , required the connection term , $h_{++}\partial_-$, as a realization of the Virasoro algebra, $diff S^1$. In the 4D self dual gravity case, the connection term is expressed in terms of Plebanski's first heavenly form, Ω and the $diff S^1$ algebra is replaced by the area-preserving diffs, which upon quantization, becomes the W_∞ algebra. The rotational Killing symmetry reduction was displayed in eqs-(2.9-2.11) which yields the dictionary between the Ω and the ρ . Such dictionary in

[25,26] will allow to infer the co-adjoint orbit quantization method of the Toda theory directly from the Killing symmetry reduction of the quantum 4D self dual gravitational effective action.

The Moyal integrable deformations of the continuous Toda equation was given by the author [19] which is suitable for the Moyal-Weyl-Wigner quantization method. q -deformed W algebras have been discussed by [54] and others. These alternative (non-conventional) quantization methods must shed some light into the operator quantization of the Toda theory. Our main objective in the next section is to establish the correspondence between highest weights states associated with the highest weights irreps of W_∞ algebras and the quantum states of the Toda molecule.

4

In this section we shall analyse in detail the U_∞ algebra obtained from a dimensional reduction of $W_\infty \oplus \bar{W}_\infty$ algebra. The highest weight representations of the W_∞ algebra [30,34,35] will be discussed and the defining relations that determine the asymptotic quantum $|\rho >_{\pm\varphi(t)}$ states in terms of the former highest weights states are explicitly furnished. The purpose of this section is to obtain important information about the highest weights representations associated with the U_∞ algebra which acts on the continuous Toda molecule as the symmetry algebra in the same way that the Virasoro algebra does for the string.

4.1. Highest Weight Representations

We can borrow now the results by [30,34,35] on the quasi-finite highest weight irreducible representations of $W_{1+\infty}$ and W_∞ algebras. The latter is a subalgebra of the former. For each highest weight state, $|\lambda >$ parametrized by a complex number λ the authors [30,34] constructed representations consisting of a finite number of states at each energy level by successive application of ladder-like operators. A suitable differential constraint on the generating function $\Delta(x)$ for the highest weights Δ_k^λ of the representations was necessary in order to ensure that, indeed, one has a finite number of states at each level. The highest weight states are defined :

$$W(z^n D^k)|\lambda > = 0. \quad n \geq 1, k \geq 0. \quad W(D^k)|\lambda > = \Delta_k^\lambda |\lambda >. \quad k \geq 0. \quad (4.1)$$

The $W_{1+\infty}$ algebras can be defined as central extensions of the Lie algebra of differential operators on the circle. $D \equiv z d/dz$. $n \in \mathbb{Z}$ and k is a positive integer. The generators of the $W_{1+\infty}$ algebra are denoted by $W(z^n D^k)$; i.e. there is a one to one mapping between $z^n (z \frac{\partial}{\partial z})^k$ and the $W_{1+\infty}$ generators. The W_∞ generators are obtained from the former : $\tilde{W}(z^n D^k) = W(z^n D^{k+1})$; where $(n, k \in \mathbb{Z}, k \geq 0)$. The commutation relations of the W_∞ are :

$$\begin{aligned} [\tilde{W}[z^n (z \frac{\partial}{\partial z})^k], \tilde{W}[z^m (z \frac{\partial}{\partial z})^l]] &= \tilde{W}[z^{n+m} (z \frac{\partial}{\partial z} + m)^k (z \frac{\partial}{\partial z})^l (z \frac{\partial}{\partial z} + m)] - \\ &\tilde{W}[z^{n+m} (z \frac{\partial}{\partial z})^k (z \frac{\partial}{\partial z} + n)^l (z \frac{\partial}{\partial z} + n)] + C \Psi(z^n (z \frac{\partial}{\partial z})^k (z \frac{\partial}{\partial z}), z^m (z \frac{\partial}{\partial z})^l (z \frac{\partial}{\partial z})). \end{aligned} \quad (4.2)$$

The central charge term is given by the two-cocycle Ψ times the constant C . (see [30,34] for further references).

The anti-chiral \bar{W}_∞ is given exactly the same by replacing everywhere $z \rightarrow \bar{z}$ and $\partial_z \rightarrow \partial_{\bar{z}}$. (There is no spin one current).

The generating function $\Delta(x)$ for the weights is :

$$\Delta(x) = \sum_{k=0}^{k=\infty} \Delta_k^\lambda \frac{x^k}{k!}. \quad (4.3)$$

where we denoted explicitly the λ dependence as a reminder that we are referring to the highest weight state $|\lambda\rangle$ and satisfies the differential equation required for quasi-finiteness :

$$b(d/dx)[(e^x - 1)\Delta(x) + C] = 0. \quad b(w) = \prod (w - \lambda_i)^{m_i}. \quad \lambda_i \neq \lambda_j. \quad (4.3)$$

$b(w)$ is the characteristic polynomial. C is the central charge and the solution is :

$$\Delta(x) = \frac{\sum_{i=1}^K p_i(x) e^{\lambda_i x} - C}{e^x - 1}. \quad (4.4)$$

The generating function for the W_∞ case is $\tilde{\Delta}(x) = (d/dx)\Delta(x)$ and the central charge is $c = -2C$.

The Verma module is spanned by the states :

$$|v_\lambda\rangle = W(z^{-n_1} D^{k_1}) W(z^{-n_2} D^{k_2}) \dots W(z^{-n_m} D^{k_m}) |\lambda\rangle. \quad (4.5)$$

The energy level is $\sum_{i=1}^{i=m} n_i$. For further details we refer to [6,7]. Highest weight unitary representations for the W_∞ algebra obtained from field realizations with central charge $c = 2$ were constructed in [30,34].

The weights associated with the highest weight state $|\lambda\rangle$ will be obtained from the expansion in (4.2). In particular, the "energy" operator acting on $|\lambda\rangle$ will be :

$$W(D)|\lambda\rangle = \Delta_1^\lambda |\lambda\rangle. \quad (4.6)$$

$L_o = -W(D)$ counts the energy level : $[L_o, W(z^n D^k)] = -n W(z^n D^k)$.

As an example we can use for $\Delta(x)$ the one obtained in the free-field realization by free fermions or **bc** ghosts [30]

$$\Delta(x) = C \frac{e^{\lambda x} - 1}{e^x - 1} \Rightarrow \partial\Delta/\partial\lambda = C \frac{x e^{\lambda x}}{e^x - 1}. \quad (4.7)$$

The second term is the generating function for the Bernoulli polynomials :

$$\frac{x e^{\lambda x}}{e^x - 1} = 1 + (\lambda - 1/2)x + (\lambda^2 - \lambda + 1/6) \frac{x^2}{2!} + (\lambda^3 - 3/2\lambda^2 + 1/2\lambda) \frac{x^3}{3!} + \dots \quad (4.8)$$

Integrating (4.8) with respect to λ yields back :

$$\Delta(x) = C \frac{e^{\lambda x} - 1}{e^x - 1} = \sum_{k=0} \Delta_k \frac{x^k}{k!}. \quad (4.9)$$

The first few weights (modulo a factor of C) are then :

$$\Delta_0 = \lambda. \Delta_1 = (1/2)(\lambda^2 - \lambda). \Delta_2 = (1/3)\lambda^3 - (1/2)\lambda^2 + (1/6)\lambda \dots \quad (4.10)$$

The generating function for the W_∞ case is $\tilde{\Delta}(x) = \frac{d\Delta(x)}{dx} \Rightarrow \tilde{\Delta}_k^\lambda = \Delta_{k+1}^\lambda$. This completes the short review of the results in [30,34]. Now we proceed to relate their construction with the results of section **III**.

4.2 The U_∞ Algebra

We are going to construct explicitly the dimensional reduction of the $W_\infty \oplus \bar{W}_\infty$ algebra, the U_∞ algebra, in terms of what one knows from the previous results in 4.1. From the previous discussion we learnt that $\tilde{\Delta}_1^\lambda = \Delta_2^\lambda$ is the weight associated with the "energy" operator. In the ordinary string, W_2 algebra, the Hamiltonian is related to the Virasoro generator, $H = L_o + \bar{L}_o$ and states are built in by applying the ladder-like operators to the highest weight state, the "vacuum". In the $W_{1+\infty}, W_\infty$ case it is *not* longer true, as we shall see, that the Hamiltonian (to be given later) can be written exactly in terms of the zero modes w.r.t the z, \bar{z} variables of the W_2 generator, once the realization of the W_∞ algebra is given in terms of the *dressed* continuous Toda field , $\Theta(z, \bar{z}, t)$, given by Savaliev [4]. The chiral generators are :

$$\tilde{W}_2^+ = \int^{t_o} dt_1 \int^{t_1} dt_2 \exp[-\Theta(z, \bar{z}; t_1)] \frac{\partial}{\partial z} \exp[\Theta(z, \bar{z}; t_1) - \Theta(z, \bar{z}; t_2)] \frac{\partial}{\partial z} \exp[\Theta(z, \bar{z}; t_2)]. \quad (4.11a)$$

$$\tilde{W}_n^+ = \int^{t_o} dt_1 \int^{t_1} dt_2 \dots \int^{t_{n-1}} dt_n \mathcal{D}_+^{(0)} \mathcal{D}_+^{(1)} \dots \mathcal{D}_+^{(n-1)} \exp[\Theta(z, \bar{z}; t_n)]. \quad (4.11b)$$

with

$$\mathcal{D}_+^{(0)} = \exp[-\Theta(z, \bar{z}; t_1)] \frac{\partial}{\partial z}; \quad \mathcal{D}_+^{(m)} \equiv \exp[\Theta(z, \bar{z}; t_m) - \Theta(z, \bar{z}; t_{m+1})] \frac{\partial}{\partial z}. \quad m \geq 1. \quad (4.12)$$

The antichiral generators are obtained upon replacing $\partial/\partial z$ by $\partial/\partial \bar{z}$ in eqs-(4.11,4.12). And now on, by continuous Toda field, one means the *dressed* continuous Toda field.

Hence, the chiral generators have the form $W_{h,0}^+[\partial^2 \rho / \partial z^2 \dots \partial^h \rho / \partial z^h]$ and the similar expression for the antichiral generators $W_{0,\bar{h}}^-$ is obtained by replacing $\partial_z \rightarrow \partial_{\bar{z}}$. After a dimensional reduction from $D = 3 \rightarrow D = 2$ is taken, $r = z + \bar{z}$, one has :

$$\tilde{W}_2(r, t_o) = \int^{t_o} dt_1 \int^{t_1} dt_2 \exp[-\rho(r, t_1)] \frac{\partial}{\partial r} \exp[\rho(r, t_1) - \rho(r, t_2)] \frac{\partial}{\partial r} \exp[\rho(r, t_2)]. \quad (4.13a)$$

And similar procedure applies to eqs-(4.11b,4.12) :

$$\tilde{W}_n = \int^{t_o} dt_1 \int^{t_1} dt_2 \dots \int^{t_{n-1}} dt_n \mathcal{D}^{(0)} \mathcal{D}^{(1)} \dots \mathcal{D}^{(n-1)} \exp[\rho(r; t_n)]. \quad (4.13b)$$

with :

$$\mathcal{D}^{(0)} = \exp[-\rho(r; t_1)] \frac{\partial}{\partial r}; \quad \mathcal{D}^{(m)} \equiv \exp[\rho(r; t_m) - \rho(r; t_{m+1})] \frac{\partial}{\partial r}. \quad (4.13c)$$

When $\rho(r, t)$ is quantized (assuming that an operator quantization can be done successfully); eqs-(4.13) involve the operator, $\hat{\rho}(r, t)$, acting on a suitable Hilbert space of states, say $|\rho\rangle$, and in order to evaluate (4.13) one needs to perform the highly complicated Operator Product Expansion between the operators $\hat{\rho}(r, t_1), \hat{\rho}(r, t_2)$. Since these are no longer free fields it is no longer trivial to compute per example the operator products :

$$\frac{\partial \rho}{\partial r} \cdot e^\rho \cdot e^{\rho(r, t_1)} \cdot e^{\rho(r, t_2)} \dots \quad (4.14)$$

It is highly nontrivial to evaluate the quantum generators, $\hat{W}_s[\rho]$. Quantization deforms the classical w_∞ algebra into W_∞ [16].

For a proof that the W_∞ algebra is the Moyal bracket deformation of the w_∞ see [17,18]. Later in [19] we were able to construct the non-linear \tilde{W}_∞ algebras from non-linear integrable deformations of Self Dual Gravity in $D = 4$.

One should not confuse ordinary Moyal deformations with Quantum-Group deformations with a natural Hopf-bialgebra structure (co-product). Quantum-Group types of W_N algebras, q - W_N , have been constructed by Awata et al and q -deformed W algebras associated with the KP hierarchy by Seco and Mas [54].

Since the w_∞ algebra has been effectively quantized the expectation value of the \tilde{W}_2 operator in the $\hbar \rightarrow 0$ limit, is related to the $\tilde{W}_2(classical)$ given by (4.12). i.e; the classical Poisson bracket algebra is retrieved by taking single contractions in the Operator Product Expansion of the quantum algebra. One can evaluate all expressions in the $r = \infty$ limit (and set $d(t) = 1$ for convenience. The expectation value in the classical limit $\langle \rho | \hat{W}_2(\hat{\rho}) | \rho \rangle = \langle \varphi(t) \rangle$ gives in the $r = \infty$ limit, after the dimensional reduction and after using the asymptotic limits :

$$\frac{\partial \rho}{\partial r} = \varphi. \quad \frac{\partial^2 \rho}{\partial r^2} = \frac{\partial^2 e^\rho}{\partial t^2} \rightarrow 0. \quad r \rightarrow \infty \quad (4.15a)$$

the value :

$$\lim_{r \rightarrow \infty} \langle \rho | \hat{W}_2 | \rho \rangle = \int^{t_o} dt_1 \varphi(t_1) \int^{t_o} dt_1 \varphi(t_1). \quad (4.15b)$$

after the normalization condition is chosen :

$$\langle \rho' | \rho \rangle = \delta(\rho' - \rho). \quad \langle \rho | \rho \rangle = 1 \quad (4.16)$$

We notice that eq-(4.15) is the same as the integrand (3.9); so integrating (4.15) with respect to t_o yields the energy as expected. Although, the explicit evaluation of the quantum generators, \hat{W}_s requires knowledge of the OPE of the Toda exponentials, one can still assume that a realization exists at the quantum level and that the asymptotic states, $|\rho(in) \rangle; |\rho(out) \rangle$ can be defined solely in terms of the (operator) $\hat{\phi}(t)$, obeying the commutation relations of (3.8b). Imagine setting $\hat{d}(t)$ to be the unit operator. The r.h.s of (3.8b) becomes zero which constrains the form of $w(t)$. Even if one wishes to avoid constraining $\hat{d}(t), w(t)$, one can still assume, based on the classical results, that the asymptotic states are purely governed by the operator $\hat{\phi}(t)$ alone. This is a reasonable assumption. The purpose becomes now to relate the $|\rho(\varphi(t) \rangle$ to the highest weight representations $|\lambda, \lambda^* \rangle$.

It is useful to recall the results from ordinary 2D conformal field theory : given the holomorphic current generator of two-dimensional conformal transformations, $T(z) = W_2(z)$, the mode expansion is :

$$W_2(z) = \sum_m W_2^m z^{-m-2} \Rightarrow W_2^m = \oint \frac{dz}{2\pi i} z^{m+2-1} W_2(z). \quad (4.17)$$

the closed integration contour encloses the origin. When the closed contour surrounds $z = \infty$. This requires performing the conformal map $z \rightarrow (1/z)$ and replacing :

$$z \rightarrow (1/z). \quad dz \rightarrow (-dz/z^2). \quad W_2(z) \rightarrow (-1/z^2)^2 W_2(1/z) = W_2(z) + \frac{c}{12} S[z', z] \quad (4.18)$$

in the integrand. $S[z', z]$ is the Schwarzian derivative of $z' = 1/z$ w.r.t the z variable.

There is also a 1 - 1 correspondence between local fields and states in the Hilbert space :

$$|\phi \rangle \leftrightarrow \lim_{z, \bar{z} \rightarrow 0} \hat{\phi}(z, \bar{z}) |0, 0 \rangle. \quad (4.19)$$

This is usually referred as the $|in \rangle$ state. A conformal transformation $z \rightarrow 1/z; \bar{z} \rightarrow 1/\bar{z}$: defines the $\langle out|$ state at $z = \infty$

$$\langle out| = \lim_{z, \bar{z} \rightarrow 0} \langle 0, 0| \hat{\phi}(1/z, 1/\bar{z}) (-1/z^2)^h (-1/\bar{z}^2)^{\bar{h}}. \quad (4.20)$$

where h, \bar{h} are the conformal weights of the field $\phi(z, \bar{z})$.

The analog of eqs-(4.20) is to consider the state parametrized by $\varphi(t), d(t)$:

$$\begin{aligned} |\rho \rangle_{\varphi(t), d(t)} &= \lim_{r \rightarrow \infty} |\rho(r, t) \rangle \equiv |\rho(out) \rangle. \\ |\rho \rangle_{-\varphi(t), d(t)} &= \lim_{r \rightarrow -\infty} |\rho(r, t) \rangle \equiv |\rho(in) \rangle. \end{aligned} \quad (4.21)$$

since the continuous Toda equation is symmetric under $r \rightarrow -r \Rightarrow \rho(-r, t)$ is also a solution and it's obtained from (3.1) by setting $\varphi \rightarrow -\varphi$ to ensure convergence at $r \rightarrow -\infty$. As we pointed out earlier, the state $|\rho \rangle$ is parametrized in terms of $\varphi(t), d(t)$ and for this reason one should always write it as $|\rho \rangle_{\varphi(t), d(t)}$. The temporal evolution of the state $|\rho \rangle$ is governed by the Hamiltonian given by eq-(5.4) in [4].

Knowing the $|\rho(in) >$ state at the time $r = -\infty$, upon radial quantization, the Hamiltonian yields the temporal evolution to another value of r .

What is required now is to establish the correspondence (a functor) between the representation space realized in terms of the continuous Toda field and that representation (the Verma module) built from the highest weight $|\lambda >$

$$< \lambda | \tilde{W}(D) | \lambda > = \tilde{\Delta}_1^\lambda \equiv \Delta_2^\lambda \leftrightarrow < \rho | \hat{W}_2[\hat{\rho}(r, t)] | \rho > . \quad (4.22)$$

If one were to extract the zero mode piece of the \tilde{W}_2 operator via a contour integral around the origin , then integrate w.r.t. the t variable and , finally, to evaluate the expectation value, one would arrive at a *trivial* result. There is a subtlety due the dimensional reduction from $2 + 1 \rightarrow 1 + 1$ dimensions. If one *naively* wrote down the expression :

$$< W_2^0 > = \int_0^{2\pi} dt_o \, {}_\varphi < \rho | [\oint \frac{dz}{2\pi i} \oint \frac{d\bar{z}}{2\pi i} \hat{W}_2(\rho(z, \bar{z}, t))] | \rho >_\varphi . \quad (4.23)$$

One now would be very hard pressed to avoid a zero answer after the contour integrations are completed in the r.h.s of (4.23). Expanding in powers of $(z + \bar{z})^n$ for negative and positive n will give a trivial zero answer for the zero-mode of the W_2 operator. If one instead expanded in positive powers of $(1/z + 1/\bar{z})^n$; i.e. lets imagine expanding the function $e^{1/z}e^{1/\bar{z}}$ containing an essential singularity at the origin , in suitable powers of z, \bar{z} , the terms containing $z^{-1}\bar{z}^{-1}$ are the ones giving the residue. But $1/z + 1/\bar{z} = (z + \bar{z})/z\bar{z} = r/z\bar{z}$ and this would contradict the assumption that all quantities depend solely on the combination of $r = z + \bar{z}$ and t . Setting $t = z\bar{z}$ is incorrect because that will constrain the original $2 + 1$ Toda theory : t is a parameter that appears in continuum graded Lie Algebras and has *nothing* to do with the z, \bar{z} coordinates. It plays the role of an extra (compact) coordinate, say an angle variable but should not be confused with the z, \bar{z} coordinates.

The correct procedure is to evaluate the generalization of what is meant by eq-(4.17). The contour integral means evaluating quantities for fixed times, which in the language of the z, \bar{z} coordinates, implies choosing circles of *fixed* radius around the origin and integrating w.r.t the angular variable. Therefore, the conserved Noether charges (the Virsoro generators in the string case) are just the integrals of the conserved currents at fixed contour-radius (fixed-times). The equal-time spatial “hypersurfaces” are then the circles of fixed radius. The expression to evaluate is no longer (4.23) but the expectation value, say w.r.t the ‘in’ state, of the zero modes of the quantity :

$$\hat{W}_2[f] = \int_0^{2\pi} dt' \, f(t') \hat{W}_2[\rho(r', t')]. \quad (4.24a)$$

where the real valued function, $f(t)$ (further details are given in eq-(4.66)) can be expanded :

$$f(t) = \sum_n a_n \cos(nt) + b_n \sin(nt). \quad (4.24b)$$

In the next subsection a detailed discussion of the value of $f(t)$ will be given. Rigorously speaking, when one writes $f(t)$ one means values at a given time, r . Further details are

presented in **4.3**. The relevant point is that $f(t)$ is not arbitrary and must obey the consistency conditions outlined in eq-(4.40).

Hence, by *zero* mode expectation value one means those w.r.t the angle variable t and *not* w.r.t the $z\bar{z}$ variables. The (zero-mode) expectation value of eq-(4.24), w.r.t the $|in\rangle$ state, is now given by :

$$\Delta_1 = \langle \hat{W}_0^{(2)}[f] \rangle_{-\varphi(t)} = \rho \left[\int_0^{2\pi} dt' a_0 \lim_{r' \rightarrow -\infty} \hat{W}_2[\hat{\rho}(r', t')] \right] | \rho \rangle_{-\varphi(t)} . \quad (4.25a)$$

$$\Delta_k = \langle \hat{W}_0^{(k)}[f] \rangle_{-\varphi(t)} = \rho \left[\int_0^{2\pi} dt' a_0 \lim_{r' \rightarrow -\infty} \hat{W}_k[\hat{\rho}(r', t')] \right] | \rho \rangle_{-\varphi(t)} . \quad (4.25b)$$

Due to the dimensional reduction, in the l.h.s of (4.25) one must take a suitable real-valued linear combination of the weights of the chiral and anti-chiral algebras. The weights live in a vector space and , as such, must be combined as vectors. First of all one imposes the conditions : $(\Delta_k^\lambda)^* = \bar{\Delta}_k^{\bar{\lambda}}$. $\bar{\lambda} = (\lambda)^*$ on all of the anti-chiral highest weights ; then one performs the linear combination of weights like those appearing in (4.10) :

$$\begin{aligned} \Delta_0 &= (\lambda + \lambda^*) ; \quad \Delta_1 = \frac{1}{2}[(\lambda + \lambda^*)^2 - (\lambda + \lambda^*)] ; \\ \Delta_2 &= \frac{1}{3}(\lambda + \lambda^*)^3 - \frac{1}{2}(\lambda + \lambda^*)^2 + \frac{1}{6}(\lambda + \lambda^*) ; \dots\dots\dots \end{aligned} \quad (4.26)$$

What one is adding as vectors are the *weights* and *not* the values of λ, λ^* . Eq-(4.26) is a particular example of how one would obtain the weights associated with the dimensional reduction of the direct sum of the chiral and anti-chiral W algebras. The most general case requires having a polynomial in $(\lambda + \lambda^*)$ of the type $\Delta_k = \frac{1}{2} \sum_{n=0}^k (a_n^k + a_n^{k*})(\lambda + \lambda^*)^n$. The explicit infinitesimal transformation law for the $\hat{\rho}(r, t)$ operator under the W_2 transformations is

$$\delta_{W_2}^{\epsilon(t)} \hat{\rho}(r, t) = \int_0^{2\pi} dt' \epsilon(t') \hat{W}_2[\hat{\rho}(r', t')] \cdot \hat{\rho}(r, t). \quad (4.27a)$$

For W_3 :

$$\delta_{W_3}^{\epsilon(t)} \hat{\rho}(r, t) = \int_0^{2\pi} dt' \epsilon(t') \hat{W}_3[\hat{\rho}(r', t')] \cdot \hat{\rho}(r, t). \quad (4.27b)$$

And for the rest :

$$\delta_{W_n}^{\epsilon(t)} \hat{\rho}(r, t) = \int_0^{2\pi} dt' \epsilon(t') \hat{W}_n[\hat{\rho}(r', t')] \cdot \hat{\rho}(r, t). \quad (4.28)$$

where the generators (after the dimensional reduction), W_2, W_3, \dots are *explicitly* given in eqs-(4.13).

Going back to eq-(4.24), the n^{th} mode component associated with the function $f(t) = \sum a_n \cos(nt) + b_n \sin(nt)$ is:

$$\hat{W}_n^{(2)}[f(t); C_r] = \int_0^{2\pi} dt [a_n \cos(nt) + b_n \sin(nt)] \hat{W}_2[\rho(r, t)]. \quad (4.29)$$

Similarly for the other generators, W_s , :

$$\hat{W}_n^{(s)}[f(t); C_r] = \int_0^{2\pi} dt' [a_n \cos(nt) + b_n \sin(nt)] \hat{W}_s[\rho(r, t)]. \quad (4.30)$$

where a radial quantization has been imposed : C_r stands for a circle of fixed radius (fixed time) r , for all values of the “angles” : t obtained by mapping the cylinder determined by the r, t variables, $-\infty \leq r \leq +\infty$ and $0 \leq t \leq 2\pi$ into the new complex plane $Z = e^{-i(t+ir)}$; $\bar{Z} = e^{i(t-ir)}$. (z, \bar{z} are *not* the same as Z, \bar{Z}).

The explicit commutation relations of the U_∞ algebra (in the realization of eqs-(4.13)) require the knowledge of the OPE of the exponentials of the Toda fields in eqs-(4.13). Shortly we will give the explicit commutation relations of U_∞ in a different realization than the Toda theory which avoids the exact knowledge of the quantization program of the Toda theory. If one has the explicit Toda field realization, for example, the equal-time (equal r) commutation relations $\hat{W}_m^{(s)}[f(t_1); C_r], \hat{W}_n^{(s')}[f(t_2); C_r]$ can be computed by performing the OPE of terms like :

$$\begin{aligned} \hat{W}_m^{(s)}[f(t_1); C_r] &= \int_0^{2\pi} dt_1 [a_m \cos(mt_1) + b_m \sin(mt_1)] \hat{W}_s[\hat{\rho}(r, t_1)] \\ \hat{W}_n^{(s')}[f(t_2); C_r] &= \int_0^{2\pi} dt_2 [a_n \cos(nt_2) + b_n \sin(nt_2)] \hat{W}_{s'}[\hat{\rho}(r, t_2)]. \end{aligned} \quad (4.31)$$

respectively. In the next subsection **4.3** we shall use our knowledge of conformal field theory methods to rewrite (4.31) in a different form. For example, in the Virasoro algebra case :

$$L_n = \oint \frac{dz'}{2\pi i} z'^{n+1} T(z'). \quad L_m = \oint \frac{dz}{2\pi i} z^{n+1} T(z). \quad (4.32a)$$

$$[L_n, L_m] = \oint \frac{dz'}{2\pi i} z'^{n+1} \oint \frac{dz}{2\pi i} z^{n+1} T(z'). T(z)$$

$$T(z'). T(z) = \frac{1}{2} \frac{c}{(z - z')^4} + \frac{2}{(z - z')^2} T(z) + \frac{1}{(z' - z)} \partial_z T(z). \quad (4.32b)$$

both of the initial contour integrals in (4.32a) enclose the origin. After the evaluation of the commutator and radial orderings; i.e after the operator product is taken , the contour is deformed analytically in such a way that the final contour integral encloses z' but *not* the origin. In this fashion one recovers the standard Virasoro algebra. Performing a change of variables $Z = e^{-i(t+ir)}$; $\bar{Z} = e^{i(t-ir)}$ will allow us to write down the U_∞ commutation relations as contour integrals in the Z, \bar{Z} plane which is *not* the same as the z, \bar{z} complex plane.

The original Lie-Poisson bracket inherent to w_∞ algebras/ area-preserving diffeomorphisms can be defined in a basis-independent way as follows $[\mathcal{L}_f, \mathcal{L}_g] = \mathcal{L}_{\{f,g\}}$. Where the infinite number of generators, \mathcal{L}_f is parametrized by a family of functions, $f(q, p)$. The infinitesimal action of \mathcal{L}_f on a function $F(q, p)$ is

$$\delta_{\mathcal{L}_f}^\epsilon F = \{\epsilon f, F\}. [\mathcal{L}_f, \mathcal{L}_g] = \mathcal{L}_{\{f,g\}} \Rightarrow [\delta_{\mathcal{L}_f}^{\epsilon^1}, \delta_{\mathcal{L}_g}^{\epsilon^2}]F = \{\{\epsilon^1 f, \epsilon^2 g\}, F\}. \quad (4.34)$$

The bracket is the ordinary Poisson bracket between two arbitrary f, g functions w.r.t the internal coordinates q, p of the two-dimensional surface, a sphere, plane, cylinder,.... per example. The area-preserving diffs of the plane is the classical w_∞ algebra. For a cylinder is the $w_{1+\infty}$ algebra and for the sphere is $su(\infty)$. Locally the algebras are isomorphic but *not* globally. If a particular basis-set of functions [16] (to expand the $f(q, p)$ functions) is chosen, like : $w_m^{(i)} = q^{m+i-1}y^{i-1}$. The i labels the particular set whereas the m labels the particular function within the particular i^{th} set. One obtains the algebra [16] :

$$\{w_m^{(i)}, w_n^{(j)}\} = [(j-1)m - (i-1)n]w_{m+n}^{(i+j-2)}. \quad (4.35)$$

The generators $w_m^{(2)}$ are the conformal-spin 2 Virasoro generators L_m and in general $w_m^{(i)}$ are the conformal spin i generators of the w_∞ algebra.

Eqs-(4.35) can be generalized to the case in where the Moyal bracket replaces the Poisson-bracket as it was shown by [17,18]. The Moyal bracket algebra deformation must not be confused with q -deformed W algebras. For a recent discussion of the importance of these algebras in q -deformed KP hierarchies see Mas and Seco [54].

It is in this fashion that the W_∞ algebra could be reconstructed. In particular, the W_∞ algebra coincides precisely with the infinite-dimensional Lie algebra of Weyl-symbols of (pseudo) differential operators on the circle S^1 . The correspondence between (pseudo) differential operators and symbols is :

$$X(\xi, z) = \sum \xi^k X_k(z) \leftrightarrow \sum X_k(z)(-i\partial_z)^k. [X, Y] \equiv X \circ Y - Y \circ X; \\ X \circ Y \equiv X(\xi, z) \exp [\partial_\xi \partial_z] Y(\xi, z). \quad (4.36)$$

where the ∂_ξ, ∂_z act on the left and right respectively. The same steps can be applied to the anti-chiral algebra counterpart, \bar{W}_∞ by replacing $z \rightarrow \bar{z}$ and $\partial_z \rightarrow \partial_{\bar{z}}$ in eqs-(4.36).

The dimensional reductions of $W_\infty \oplus \bar{W}_\infty$ will reproduce automatically the U_∞ algebra in a realization which *differs* from the Toda one. The dimensionally reduced U_∞ algebra is defined directly from eq-(4.2) by setting $z, \bar{z} \rightarrow r$ and $\partial_z, \partial_{\bar{z}} \rightarrow \partial_r$. The generators of the U_∞ algebra are denoted by $W_n^{(k)} \leftrightarrow \tilde{W}(r^n(r\frac{\partial}{\partial r})^k)$ and the latter generate a subalgebra of (the dimensionally reduced) $W_{1+\infty}$ algebra whose generators are denoted by $W(z^n D^k); D \equiv z\partial/\partial z$, by choosing : $\tilde{W}(r^n D^k) \equiv W(r^n D^{k+1})$ for n, k integers and $k \geq 0$. The U_∞ algebra reads :

$$[\tilde{W}[r^n(r\frac{\partial}{\partial r})^k], \tilde{W}[r^m(r\frac{\partial}{\partial r})^l]] = \tilde{W}[r^{n+m}(r\frac{\partial}{\partial r} + m)^k(r\frac{\partial}{\partial r})^l(r\frac{\partial}{\partial r} + m)] -$$

$$\tilde{W}[r^{n+m}(r\frac{\partial}{\partial r})^k(r\frac{\partial}{\partial r}+n)^l(r\frac{\partial}{\partial r}+n)] + C\Psi(r^n(r\frac{\partial}{\partial r})^k(r\frac{\partial}{\partial r}), r^m(r\frac{\partial}{\partial r})^l(r\frac{\partial}{\partial r})). \quad (4.37)$$

The 2-cocycle, Ψ is defined in [30,34] and C is the central charge. At the group level there are some subtleties involved due to the fact that the circle is not the same as the real line.

Eq-(4.37) is obtained abstractly (without the need to quantize the Toda theory). The one-to-one mapping to the Toda field realizations yields the equal-time (fixed r) commutation relations. First, one has :

$$\tilde{W}(r^n(r\frac{\partial}{\partial r})^k) \leftrightarrow W_n^s[\rho(r, t)]; \quad k = 1, 2, 3, \dots \quad s = k + 1 = 2, 3, 4, \dots \quad (4.38a)$$

where the explicit form of the generators $W_s[\rho]$ was given in eqs-(4.13) and the n^{th} modes by eqs-(4.29,4.30). The commutators are :

$$[\tilde{W}_n^k, \tilde{W}_m^l] = [\tilde{W}(r^n(r\frac{\partial}{\partial r})^k), \tilde{W}(r^m(r\frac{\partial}{\partial r})^l)] \leftrightarrow [W_n^s[\rho(r, t)], W_m^{s'}[\rho(r, t)]]. \quad (4.38b)$$

for $n, l = 0, 1, 2, \dots$. Eventhough one cannot evaluate, explicitly, the commutation relations (4.38) until one has a grasp of the full operator quantization program of the continuous Toda theory with the OPE of the exponentials of the Toda field and its associated Quantum Group structures, Exchange Algebra..., one can still bypass the latter quantization process, and read-off the terms appearing in the commutators of eq-(4.38) :

$$[\tilde{W}_n^k, \tilde{W}_m^l] = \sum_{p=0}^{\infty} g_{2p}^{kl}(n, m) \tilde{W}_{n+m}^{k+l-2-2p} + c\delta^{kl}\delta_{n+m,0} \frac{1}{k-1} (C_{k-1}^{2(k-1)})^{-1} (C_k^{2k})^{-1} \prod_{j=-(k-1)}^{k-1} (n+j). \quad (4.39)$$

The r.h.s involves binomial combinatoric coefficients, C_l^k and hypergeometric functions for the term $g_{2p}^{kl}(n, m)$. For further details we refer to [30,34].

Thus, the U_{∞} commutators can be read-off from the correspondence (after the dimensional reduction) :

$$[W_n^s[\rho(r, t)], W_m^{s'}[\rho(r, t)]] \leftrightarrow [\tilde{W}_n^k, \tilde{W}_m^l]. \quad \tilde{W}_n^k \leftrightarrow W_n^{s=k+1}[\rho(r, t)]. \quad \tilde{W}_m^l \leftrightarrow W_m^{s'=l+1}[\rho(r, t)]. \quad (4.40a)$$

and the r.h.s of the commutators can be read-off from the r.h.s of (4.39):

$$\tilde{W}_{n+m}^{(k+l-2-2p)} \leftrightarrow \hat{W}_{n+m}^{(k+l-2-2p+1)}[\hat{\rho}(r, t)] = \int dt (a_{n+m} \cos[(n+m)t] + b_{n+m} \sin[(n+m)t]) \hat{W}_s[\hat{\rho}(r, t)]. \quad s = k + l - 2 - 2p + 1. \quad (4.40b)$$

The set of eqs-(4.37-4.40) *define* the U_∞ algebra in the realization furnished by eqs-(4.13) and clearly restrict the values the function $f(t)$ can take. The latter eqs-(4.40) constrain explicitly the value of $f(t)$. Eqs-(4.63-4.66), below, will determine the $f(t)$ in a different coordinate basis. The U_∞ algebra is defined without the need to evaluate explicitly the generators, W_s , in terms of the quantized field operator : $\hat{\rho}$. Secondly, for every highest weight representation of the original W_∞, \bar{W}_∞ algebras constructed by [30,34,35] one can always write down the corresponding weights explicitly (after following the dimensional reduction procedure of eqs- (4.26)). Upon inserting the values of the weights into the l.h.s of eqs-(4.25), the latter equations can then be used to *define*, in an implicit and indirect manner, the corresponding $|\rho(in/out) >$ state in terms of the spectral decomposition of the operator $\hat{\varphi}((\lambda + \lambda^*), t)$. The converse is not true. For every asymptotic quantum state $|\rho(in/out) >$, appearing in the r.h.s of (4.25) one cannot claim that it always corresponds to a particular highest weight representation. The correspondence is not necessarily 1 – 1. Eqs-(4.25) are the *defining* relations which determine, in principle, the correspondence between the highest weights $|\lambda, \lambda^* >$ and $|\rho(in/out) >_{\mp\varphi(t)}$.

To finalize, in the remainder of this subsection, we will describe how the expectation values of the W_∞ generators w.r.t the “in” states are taken. Firstly, if one is taking expectation values in the “in” states one has to evaluate the $\lim r' \rightarrow -\infty$ of the \hat{W}_s generators. The latter generators are expressed in terms of the quantum solutions of the continuous Toda theory. Secondly, setting aside for the moment the singularities appearing in the OPE of Toda field exponentials, prior to taking the $r' = -\infty$ limit, one has after evaluating (4.25) expressions of the type :

$$\begin{aligned} -_{\varphi(t)} < \rho | \hat{I}(t = 2\pi) - \hat{I}(t = 0) | \rho >_{-\varphi(t)} = -_{\varphi(t=2\pi)} < \rho | \hat{I}(t = 2\pi) | \rho >_{-\varphi(t=2\pi)} - \\ -_{\varphi(t=0)} < \rho | \hat{I}(t = 0) | \rho >_{-\varphi(t=0)} = I(2\pi) - I(0). \end{aligned} \quad (4.41)$$

after using the normalization condition $< \rho | \rho > = 1$. Thirdly, modulo the subtleties in pulling the expectation value inside the t' -integration, one can verify that the same answer is obtained if one performs the expectation value before the t' integration . One evaluates the action of an operator of ρ acting on the “in” state as follows :

$$\mathcal{F}[\hat{\rho}(r', t')] | \rho >_{-\varphi(t)} = \lim_{r' \rightarrow -\infty, t' \rightarrow t} F[\rho(r', t')] | \rho >_{-\varphi(t)} . \quad (4.42)$$

By inspection one can verify that after performing the integration one arrives at the same result of (4.41) . This is the analog of $f[\hat{x}] | x' > = f(x') | x' >$. The integral (4.41) clearly diverges as a result of the OPE of the Toda exponentials prior to taking the $r' = -\infty$ limit. This can be fixed by introducing a “normal ordering” procedure that removes the short distance singularities as a result of the coincidence limit of two or more operators. What is important is the t integration. Since the l.h.s of (4.25) has a dependence on the $\lambda, \bar{\lambda}$ parameters the $\hat{\varphi}(t)$ must also encode such a dependence. The operator $\hat{\varphi}(t)$ can be expanded :

$$\hat{\varphi}(\lambda, \bar{\lambda})(t) = \sum_n \hat{A}_n(\lambda, \bar{\lambda}) \cos(nt) + \hat{B}_n(\lambda, \bar{\lambda}) \sin(nt). \quad (4.43)$$

plugging (4.43) into (4.25) yields the infinite number of equations with an explicit $\lambda, \bar{\lambda}$ dependence. The infinite number of integrals like (4.25) involving the weights associated with the zero modes of the algebra generators are :

$$\Delta_k^{\lambda+\lambda^*} =_{-\varphi(t)} \rho \left[\int_0^{2\pi} dt' a_0 \hat{W}_s [\hat{\rho}(r' = -\infty, t')] \right] | \rho >_{-\varphi(t)} . \quad (4.44)$$

with $k = 1, 2, 3, 4, \dots, \infty$ and $s = k + 1 = 2, 3, 4, \dots, \infty$ and the operator-coefficient a_0 will be determined in 4.3 to be $1/2\pi$.

The weights , after the dimensional reduction, are in general of the form :

$$\Delta_k = \frac{1}{2} \sum_{n=0}^k (a_n^k + a_n^{*k}) (\lambda + \lambda^*)^n. \quad (4.45)$$

The weights are polynomials in $(\lambda + \lambda^*)$. Choosing the eigenvalues of \hat{A}_n, \hat{B}_n in (4.43) to be polynomials of order n in $\lambda + \lambda^*$ yields : $A_n = P_n(\lambda + \lambda^*)$, and $B_n = P_n(\lambda + \lambda^*)$; this will be sufficient to solve for (4.25, 4.32).

A real-valued polynomial of order n has $n + 1$ independent coefficients. Say one opts to choose : $A_n = a_0^n + a_1^n(\lambda + \lambda^*) + \dots + a_n^n(\lambda + \lambda^*)^n$. Each real-valued polynomial of order n has $n + 1$ independent coefficients, thus, the total number of independent coefficients in the infinite number of polynomials is of order $2 \sum_{n=0}^{\infty} (n + 1) \sim n^2$, where one has included those coefficients stemming from the B_n terms as well. Now, the l.h.s of (4.44) generates k equations, each having the weight-polynomials of degree $k + 1$ in $\lambda + \lambda^*$ containing $(k + 2)$ known coefficients; so the total number of known coefficients is $\sum_1^{\infty} (k + 2) \sim k^2/2$. The rate of growth in the number of independent coefficients is the same in both cases. To have an exact match in the number of equations and in the number of independent variables one requires to impose a constraint on the polynomials A_n, B_n : $B_n = 0$, for example, and the doubling problem is eliminated.

Therefore, by matching, level by level in k , the coefficients involved in the powers of $(\lambda + \lambda^*)^0, (\lambda + \lambda^*)^1, (\lambda + \lambda^*)^2, \dots$ one can solve for the A_n polynomials in the variable $\lambda + \lambda^*$ and determine the (expectation value of) $\hat{\varphi}(t)$ uniquely in (4.43).

Hence , an explicit knowledge of the weights associated with a given representation fixes the form of the eigenvalues of the operator $\hat{\varphi}(t)$ and furnishes the asymptotic $|\rho(in) >$ state uniquely in terms of the $\lambda + \lambda^*$ parameter. The same applies to the $|\rho(out) >$ state. Thus, the family of quantum states of the putative quantum Toda molecule theory associated with the highest weight representations of the dimensionally reduced U_{∞} algebra, can be explicitly classified. There many other states which correspond to other representations.

To recapitulate : The spectrum generating U_{∞} symmetry algebra acting on the Toda molecule, stems from the bosonic sector of the self-dual $SU(\infty)$ Supersymmetric Gauge Quantum Mechanical Model associated with the light-cone gauge of the self-dual (spherical) supermembrane : a dimensionally-reduced super SDYM theory to *one* temporal dimension. The membrane's time coordinate (X^+) has a *correspondence* with the $r = z + \bar{z}$ variable. The extra coordinate arises from the t parameter so the initial 3D continuous (super) Toda theory is dim-reduced to a 1+1 (super) Toda *molecule* : $\Theta(z, \bar{z}, t) \rightarrow \rho(r, t)$ and, in this way, an *effective* two-dimensional theory emerges. Hence, the intrinsic 3D

Self Dual membrane spectrum can be obtained from the U_∞ symmetry algebra (spectrum generating algebra) of the effective two-dimensional (super) Toda molecule theory. The full spectrum remains an open question. Perhaps (nonlinear) integrable deformations beyond the selfdual theories (non-conformal field theories) might give us more clues about the full theory.

4.3 The U_∞ Algebra in the Z, \bar{Z} Representation

So far we have worked with integrals involving the $r = z + \bar{z}, t$ variables. One could define the *new* complex variables exploiting the fact that t is a *periodic* variable parametrizing a circle [4]:

$$U \equiv t + ir, \bar{U} \equiv t - ir, Z = e^{-iU} = e^r e^{-it}, \bar{Z} = e^r e^{it} \quad (4.46)$$

to obtain a new complex domain Z, \bar{Z} from the cylinder defined by the U coordinates such that $r = -\infty$ corresponds to the origin $Z, \bar{Z} = 0$ and $r = +\infty$ corresponds to infinity. In this fashion a Hamiltonian analysis based on r quantization (the original time coordinate) corresponds to a “radial” quantization in the complex plane as conventional CFT requires.

It is precisely when this correspondence is made that the two-dimensional world of non-critical W_∞ strings matches the two effective dimensions which follow the dimensional reduction from the $2 + 1$ continuous Toda theory to $1 + 1$ dimensions ; that of the r, t variables. It is in this fashion how it makes sense to claim that non-critical W_∞ strings in $D = 27$ are connected to critical bosonic membranes in $D = 27$ (after the light-cone gauge is chosen so a $SU(\infty)$ YM theory dimensionally reduced to one temporal dimension emerges). Both theories are effectively two-dimensional.

One should not worry at the moment about the fact that the area-preserving-diffs of the cylinder is $w_{1+\infty}$ algebra instead of the w_∞ algebra associated with the plane. The presence of the w_∞ structures originate from the $3D$ continuous Toda theory. The physically relevant area-preserving-diffs algebra which one is always referring to is the $su(\infty)$ algebra associated with the sphere.

It is important to point out that the generator $W_2[\rho(r, t)]$ (4.13) obtained as a dimensional reduction of (4.12) (and its antichiral counterpart) is a *mixed* tensor w.r.t the Z, \bar{Z} coordinates. This is easily derived by starting with the integral of $W_2[\rho(r, t)]$ w.r.t the “angle” variable t to yield the energy. The effective two dimensional space with coordinates, r, t , is a cylinder. r plays the role of time and t of space (an angle). Hence the conserved quantity w.r.t. the timelike Killing vector, $v^b = v^r(\partial/\partial r)$, is the energy given by the integral over a spacelike surface Σ^a (whose oriented normal is timelike) of the quantity : $T_{ab}v^b$:

$$\mathcal{H} = \int dt W_2[\rho(r, t)] \equiv \int d\Sigma^a T_{ab}v^b = \int dt T_{rr} = Energy. \quad (4.47)$$

where $d\Sigma^r = dt; v^r = 1$. Because $T_{rr}(r, t) \equiv W_2[\rho(r, t)]$ is one component of the stress energy tensor it must transform under the coordinate transformations : $r \rightarrow \frac{1}{2}\ln(Z\bar{Z}), t \rightarrow \frac{1}{2}\ln(Z/\bar{Z})^i$ as follows :

$$T_{rr}(r, t) = W_2[\rho(r, t)] = \frac{\partial Z}{\partial r} \frac{\partial Z}{\partial r} W_{ZZ}(Z, \bar{Z}) + \frac{\partial \bar{Z}}{\partial r} \frac{\partial \bar{Z}}{\partial r} W_{\bar{Z}\bar{Z}}(Z, \bar{Z}) +$$

$$\frac{\partial Z}{\partial r} \frac{\partial \bar{Z}}{\partial r} W_{Z\bar{Z}}(Z, \bar{Z}) + \frac{\partial \bar{Z}}{\partial r} \frac{\partial Z}{\partial r} W_{\bar{Z}Z}(Z, \bar{Z}). \quad (4.48)$$

At the quantum level there might be anomalies in the transformation laws. For example, when one maps the cylinder into the punctured complex plane by a conformal mapping, there is a central charge contribution to the L_o Virasoro generator arising from the Schwarzian derivative $T(z) \rightarrow (\partial_z f)^2 T[f(z)] + (c/12)S(f(z), z)$. At the moment we shall not be concerned with anomalies since the $D = 11$ supermembrane is expected to be anomaly free. Central charges will emerge in the OPE of the generators, W_s .

An important remark is in order. Let us assume knowing the remaining components T_{rt}, T_{tt}, T_{tr} of the stress energy tensor in addition to $T_{rr} \equiv W_2[\rho(r, t)]$. In terms of the new complex coordinates a new tensor is constructed with components $:W_{ZZ}; W_{\bar{Z}\bar{Z}} \dots$. Does this stress energy tensor generate new holomorphic/antiholomorphic transformations? $Z \rightarrow Z'(Z); \bar{Z} \rightarrow \bar{Z}'(\bar{Z})$ The answer is *no*. By inspection we can see, for example, that under dilatations $z'(z) = \lambda z; \bar{z}'(\bar{z}) = \lambda \bar{z}$ one does not generate $Z'(Z) = \Lambda Z; \bar{Z}'(\bar{Z}) = \Lambda \bar{Z}$. And viceversa, in general $Z \rightarrow F(Z), \bar{Z} \rightarrow \tilde{F}(\bar{Z})$ does not always imply $z' = f(z); \bar{z}' = \tilde{f}(\bar{z})$. Thus, the components of the new stress energy tensor cannot longer be split in terms of two *independent* holomorphic/antiholomorphic pieces: $T(Z) = W_{(2,0)}^+(Z), \tilde{T}(\bar{Z}) = W_{(0,2)}^-(\bar{Z})$ as it occurs in ordinary CFT. The dimensional reduction *intertwines* the holomorphic/antiholomorphic variables and the U_∞ algebra does *not* generate conformal transformations in the Z, \bar{Z} variables.

We can write down eqs-(3.9,4.15) again for the classical generator W_2 and by inspection see that :

$$W_2[\rho(r, t)] \sim \left(\int^t dt' \varphi(t') \right)^2 = F(t) = F[(\ln(Z/\bar{Z}))^i] \neq f(Z) + \bar{f}(\bar{Z}). \quad (4.49)$$

A splitting cannot occur unless a very special choice of $\varphi(t)$ is chosen. There is no reason why $F(U + \bar{U}) = f(U) + \bar{f}(\bar{U})$ unless one has a linear function.

Let us define a meromorphic real-valued field operator of two complex variables, Z, \bar{Z} , $\hat{W}_2(Z, \bar{Z})$, to be determined in terms of $W_2[\rho(r, t)]$ admitting the following expansion in powers of Z, \bar{Z} :

$$\hat{W}_2(Z, \bar{Z}) = \sum_{N, \bar{N}} \hat{W}_2^{N, \bar{N}} Z^{-(n+im)-1} \bar{Z}^{-(n-im)-1}. \quad (4.50)$$

where $N \equiv n + im$. $\bar{N} \equiv n - im$. The mode expansion in the Z, \bar{Z} variables (4.50) *differs* from the expansion w.r.t the z, \bar{z} variables as a result of the fact that the generator $W_2[\rho(r, t)] = T_{rr}(r, t)$ is a particular component of a *mixed* stress energy tensor that intertwines z, \bar{z} . This is to be expected. Therefore, the correct expression to extract the residues is :

$$\Delta_1 = \langle \rho(in) | \hat{W}_2^{N=0, \bar{N}=0} | \rho(in) \rangle = \langle \rho(in) | \left[\oint \frac{dZ}{2\pi i} \oint \frac{d\bar{Z}}{2\pi i} \hat{W}_2(Z, \bar{Z}) \right] | \rho(in) \rangle. \quad (4.51)$$

If (4.51) is equal to (4.25) then certain conditions must be met . If we integrate (4.51) around circles of fixed-radius $R = e^r$ (and then take the $R \rightarrow 0$ limit) one has after using : $dZ = -iZdt; d\bar{Z} = i\bar{Z}dt$ and taking into account an extra minus sign due to the counterclockwise/clockwise Z, \bar{Z} contour integrations the following condition :

$$\int_0^{2\pi} \frac{dt}{2\pi} \int_0^{2\pi} \frac{dt}{2\pi} \lim_{R \rightarrow 0} R^2 \hat{W}_2(Re^{-it}, Re^{it}) = \int_0^{2\pi} dt a_0 W_2[\rho(r = -\infty, t)]. \quad (4.52)$$

hence one learns from (4.48) and (4.52) that :

$$a_0 = \frac{1}{2\pi}. W_2[\rho(r, t)] = Z\bar{Z}\hat{W}_2(Z, \bar{Z}) = Z\bar{Z}(W_{Z\bar{Z}} + W_{\bar{Z}Z}). Z^2W_{ZZ} + \bar{Z}^2W_{\bar{Z}\bar{Z}} = 0. \quad (4.53)$$

Later we shall study the conditions on the $f(t) = \sum a_n \cos(nt) + b_n \sin(nt)$ in order to find the relations amongst the higher order modes and for all the W_s generators in addition to the W_2 .

Despite the fact that one has no longer purely holomorphic/antiholomorphic transformations, the transformations generated by the U_∞ generators $\hat{W}_2(Z, \bar{Z})$ acting on $\tilde{\rho}(Z, \bar{Z})$ can, nevertheless, be written as :

$$\delta_{W_s}^\epsilon \tilde{\rho} = \oint \oint \frac{dZ}{2\pi i} \frac{d\bar{Z}}{2\pi i} \epsilon(Z, \bar{Z}) \hat{W}_s(Z, \bar{Z}) \tilde{\rho}(Z', \bar{Z}'). \quad (4.54)$$

after using the standard techniques of OPE and contour deformations of CFT. The contours surround the Z', \bar{Z}' variables and *not* the origin after the contour deformation is performed. The U_∞ algebra obtained from the dimensional-reduction process of the W_∞ extended conformal field theory inherits a similar transformation structure.

Focusing on (4.51), the $|\rho(in) >_{-\varphi(t)}$ state at $r = -\infty$ is now the state corresponding to the operator insertion at the origin of the punctured Z, \bar{Z} plane. To define the “in” state requires evaluating the limit :

$$|\lambda, \bar{\lambda} > \leftrightarrow \lim_{Z, \bar{Z} \rightarrow 0} r(Z, \bar{Z}) \hat{\phi}_{\lambda, \bar{\lambda}}[t(Z, \bar{Z})] |0, 0 >. \quad (4.55)$$

The operator quantity $-\hat{\phi}(t)$ appearing in the quantum Toda solutions bears an explicit $\lambda, \bar{\lambda}$ dependence (to be determined below) and defines the $|in >$ state at $r = -\infty$ in eq-(4.21). Now it makes sense to evaluate the contour integrals using the residue theorem without obtaining a trivial zero answer. If one opts to perform the contour integration *before* the expectation value is taken in (4.51) is more convenient to take a contour surrounding the *origin* which will absorb the singularities of the OPE in the coincidence limits $Z', \bar{Z}' \rightarrow Z, \bar{Z} \rightarrow 0$. In general the contour integration does *not* necessarily commute with the evaluation process of taking the expectation value :

$$< W_2^{0,0} > \sim \oint \frac{dZ}{2\pi i} \oint \frac{d\bar{Z}}{2\pi i} < \rho_o | [\hat{W}_2((Z, \bar{Z}))] | \rho_o >. \quad (4.56)$$

because one cannot naively pull out the contour integrals outside the expectation values without introducing contact terms due to the time-ordering procedures (in the expectation values) leading to delta function singularities. Setting these subtleties aside, integrals (4.44,4.56) are roughly the same.

In (4.51) the states $|\rho(in) \rangle$ are the sought-after highest weight states (ground states). The physical states belong to a particular class of the former. In order to obtain the space of physical states a complete list of all *unitary* highest weight irrerepresentations is required and from these a no-ghost theorem can be formulated which will select the restricted values of the infinite number of conformal weights , Δ_k , as well as selecting the value of the critical central charges (or spacetime dimension as it occurs for the string). This is tantamount of writing down the complete BRST cohomology of the physical vertex operators linked to the (dimensionally-reduced) W_∞ CFT. This is a extremely arduous task because we don't have a W_∞ CFT. Ordinary string theory is based on a W_2 CFT, rational or irrational CFT, and we know how difficult matters can be. From each of these sought-after highest weight states one builds a tower (the Verma module) by successive applications of the ladder like operators (4.5). In ordinary string theory the tower of states are the so called spurious states. These are *physical* when $D = 26, a = 1$ because the norm is zero. Eqs-(4.1) are the analog of the string mass shell condition $:(L_o - a)|\phi \rangle = 0$ where $a = 1$ results from the zeta function regularization of the string zero-point-energy states. This is the reason why the expectation values of the zero mode operators $W_2^{N,\bar{N}}$ would require regularization as well.

If the evaluation of (4.51,4.56) displays singularities in the operator product the energy is ill-defined. Therefore, a regularization is required so that the expectation value of (4.51,4.56) in the quantum case is finite. In ordinary CFT this is achieved after a conformal normal ordering procedure (or other suitable ordering) is introduced so that correlations functions are finite and obey the Ward identities. The conformal normal ordering removes in most cases the infinities. It is customary to define the normal ordered product of two field-operators, $(\mathcal{A}\mathcal{B})(w)$ as a contour integral surrounding w of $(z - w)^{-1}\mathcal{A}(z)\mathcal{B}(w)$. It is essentially a point-splitting regularization method. The normal ordering does not obey the commutative nor associative property.

The regularization method involves the point-splitting method discussed above. $\varphi(t)$ must now be interpreted as the expectation value (to be determined shortly) of the $\hat{\varphi}(t)$ operator which appears in the putative quantum solutions (given by eqs-(3.1,3.6,3.7). These asymptotic expressions are given in terms of eq-(3.4) where now $\varphi(t), \ln d(t)$ are operators obeying the commutation relations (3.8b). As stated earlier we have assumed that the asymptotics states depend solely on $\varphi(t)$.

The state $|\rho \rangle$ in general could be any state in the Hilbert space of states associated with the operator $\hat{\rho}(r, t)$ for any value of r ; i.e. it is the state associated with the *interacting* highly nonlinear Toda field. At the moment we don't know if there is a 1-1 correspondence between local fields inserted at particular locations in the punctured complex plane (or other Riemann surface) and states in this dimensionally-reduced W_∞ CFT, as it happens with ordinary CFT. Choosing the asymptotic states simplify matters considerably because the fields become free. The asymptotic limit was uniquely determined in eq-(3.4) by the operator $\hat{\rho}_o = (\partial^2 \hat{x}_o / \partial t^2) \rightarrow r \hat{\varphi}(t)$.

A question comes to mind : “Where does the \hbar -dependence of the energy (given by the integral of eq-(4.15) w.r.t the t variable) come from ?” In eqs-(3.6,3.7) the quantization was encoded, for the most part, in the function $w(t)$ which yields the $O(\hbar)$ corrections to the $\sum \varphi(t)^{-1}$ terms in eq-(3.1). In this Heisenberg representation the quantum solutions for $\rho(r, t)$ are seen as operator valued quantities with an *explicit* \hbar dependence in (3.6,3.7). It is for this reason that $\varphi(t)$ must be taken to be an operator (as well as $d(t)$). After expectation values are taken in the asymptotic limits the explicit \hbar factors appear in the eigenvalues of the operator $\hat{\varphi}(t)$ that carry the \hbar dependence. Whence the presence of the \hbar . ($d(t)$ was chosen earlier to be the unit operator, this might create problems in the quantization, the commutator of (3.8b) degenerates to zero). The eigenvalue equation needed to compute the expectation values in the $|\rho(in) >$ states reads :

$$|\rho_{\lambda, \bar{\lambda}} > = \lim_{Z, \bar{Z} \rightarrow 0} r(Z, \bar{Z}) \hat{\varphi}_{\lambda, \bar{\lambda}}[t(Z, \bar{Z})] |0, 0 >. \quad (4.57)$$

Therefore, the \hbar dependence is encoded in the eigenvalues/ weights of the $\hat{\varphi}(t)$ operator. For example, the angular momentum states are $|J, J_z >$ such that $\hat{J}_z |J, J_z > = m_z \hbar |J, J_z >$. One could set the r.h.s of (3-8b) to one so now $\varphi(t), lnd(t)$ behave like canonical conjugate operators.

Integrals like (4.51) are not the only ones required to determine $\varphi_{\lambda, \bar{\lambda}}(t)$ To find such an explicit $\lambda, \bar{\lambda}$ dependence of $\varphi(t)$ one needs to recur to *all* the weights of the representation, Δ_k^λ (and the anti-chiral ones). These are also related to the zero modes of Saveliev’s realization of the chiral and antichiral W_∞ algebras in terms of the dressed continuous Toda field [4], where the infinite number of generators have similar form to eq-(4.13) (the number of t integrations depends on the value of the conformal spin) :

$$\mathcal{W}_{(h,0)}^+[\partial^2 \rho / \partial z; \dots \partial^h \rho / \partial z^h]. \mathcal{W}_{(0,\bar{h})}^-[\partial_z \rho \rightarrow \partial_{\bar{z}} \rho]. \partial \mathcal{W}^+ / \partial \bar{z} = 0. \partial \mathcal{W}^- / \partial z = 0 \quad (4.58)$$

After the dimensional reduction is taken, the expectation values of the zero modes, expressed in terms of the new variables, Z, \bar{Z} , of the U_∞ generators are :

$$< |\hat{W}_s^{N=0, \bar{N}=0} > = < \rho(in) | \left[\oint \frac{dZ}{2\pi i} \oint \frac{d\bar{Z}}{2\pi i} Z^{\frac{s}{2}-1} \bar{Z}^{\frac{s}{2}-1} \hat{W}_s(Z, \bar{Z}) \right] | \rho(in) > = \Delta_k. \quad (4.59)$$

where $k \geq 1$, $s = k + 1 = 2, 3, 4, \dots, \infty$. The real-valued meromorphic field operator, $\hat{W}_s(Z, \bar{Z})$, admits an expansion similar to (4.50) in powers of $Z^{-N-s/2} \bar{Z}^{-\bar{N}-s/2}$. Having in some instances half-integer exponents in (4.59) is not that harmful, these are very natural in the fermionic string.

If eqs-(4.44) are to be equal to (4.59) similar conditions like (4.52) must be met. After performing the contour integrals around the origin by means of circles of fixed radius as it was done in (4.52) , using $Z = Re^{-it}$, $\bar{Z} = Re^{it}$, $R = e^r$, yields :

$$\int_0^{2\pi} \frac{dt}{2\pi} \lim_{R^2 \rightarrow 0} (R^2)^{\frac{s}{2}} \hat{W}_s(Re^{-it}, Re^{it}) = \int_0^{2\pi} dt a_0 W_s[\rho(-\infty, t)]. \quad (4.60)$$

this occurs for fixed radius, $R = e^r$, hence one learns that in general one must have :

$$a_0 = \frac{1}{2\pi} \cdot (Z\bar{Z})^{s/2} \hat{W}_s(Z, \bar{Z}) = W_s[\rho(r, t)]. \quad (4.61)$$

The conditions on the other higher modes can be met also if the n^{th} component, $f_n(t)$, of the function $f(t)$ obeys the following equation :

$$\int_0^{2\pi} dt f_n(t) W_s[\rho(-\infty, t)] = \int_0^{2\pi} \frac{dt}{2\pi} \lim_{R \rightarrow 0} (R^2)^{s/2} (R^2)^n (Z/\bar{Z})^{im} \hat{W}_s(Re^{-it}, Re^{it}). \quad (4.62)$$

where one t integration absorbs a 2π factor.

An equality can be obtained iff $m = \frac{n}{2} \Rightarrow N \equiv n(1 + i/2); \bar{N} \equiv n(1 - i/2)$ and the $f(t)$ admits the expansion in a different basis, $\cosh(nt); \sinh(nt)$ instead of cosines/sines : $f(t) = \sum_n a_n \cosh(nt) + b_n \sinh(nt)$. Plugging $f_n(t)$ in (4.62) yields after matching term by term in n :

$$\begin{aligned} \lim_{R \rightarrow 0} (R^2)^{s/2} \hat{W}_s(Re^{-it}, Re^{it}) &= W_s[\rho(-\infty, t)]. \\ a_n \cosh(nt) + b_n \sinh(nt) &= \lim_{R \rightarrow 0} \frac{R^{2n} e^{nt}}{2\pi}. \end{aligned} \quad (4.63)$$

If the last relation holds at a *fixed* value of $R = e^r$, its generalization to other values of Z, \bar{Z} would require incorporating an r dependence to the original $f(t)$ function. This implies that one must have for the a_n, b_n coefficients an explicit r dependence : $a_n(r); b_n(r)$ in order for (4.63) to be consistent. The condition (valid for other values of Z, \bar{Z}) is

$$(Z\bar{Z})^{s/2} \hat{W}_s(Z, \bar{Z}) = W_s[\rho(r, t)]. \quad a_n(r) = \tilde{a}_n e^{2nr}. \quad b_n(r) = \tilde{b}_n e^{2nr}. \quad (4.64)$$

Eq-(4.63) yields :

$$\tilde{a}_n \equiv \lim_{R \rightarrow 0} \frac{a_n(r = \ln R)}{R^{2n}}, \quad \tilde{b}_n \equiv \lim_{R \rightarrow 0} \frac{b_n(r = \ln R)}{R^{2n}}. \quad (4.65)$$

so that the $R = e^r$ factors cancel out leaving :

$$\tilde{a}_n \cosh(nt) + \tilde{b}_n \sinh(nt) = \frac{e^{nt}}{2\pi}. \quad (4.66)$$

in agreement with the previous conclusion that $a_0 = \tilde{a}_0 = (1/2\pi)$.

Concluding, we have that the real-valued meromorphic field $\hat{W}_s(Z, \bar{Z})$ is related to $W_s[\rho(r, t)] = \tilde{W}_s[\tilde{\rho}(Z, \bar{Z})]$ as follows :

$$(Z\bar{Z})^{s/2} \hat{W}_s(Z, \bar{Z}) = W_s[\rho(r, t)] = \tilde{W}_s[\tilde{\rho}(Z, \bar{Z})]. \quad (4.67a)$$

: with the proviso that $f(t, r)$ admits the expansion with r dependent coefficients shown above in eq-(4-64, 4.66) and :

$$\hat{W}_s(Z, \bar{Z}) = \sum_{N, \bar{N}} \hat{W}_s^{N, \bar{N}} Z^{-N-s/2} \bar{Z}^{-\bar{N}-s/2}. \quad N = n + in/2; \bar{N} = n - in/2. \quad (4.67b)$$

As remarked earlier, eqs-(4.40), clearly constrain the form of $f(t)$. To solve the infinite number of consistency conditions given by eqs-(4.40), is very difficult. The point in recurring to eq-(4.63-4.66) is to show how restricted $f(t)$ must be in a given basis. Had we use the r, t coordinates instead of Z, \bar{Z} a different representation for $f(t)$ would have been obtained.

Eq-(4.67b) is a realization of the real-valued meromorphic field operators in terms of solutions of the continuous Toda molecule ; $\rho(r, t)$ after the change of variables is performed. It is clear from (4.67a) that the meromorphic fields are mixed and are no longer purely holomorphic nor antiholomorphic in the Z, \bar{Z} variables.

To define the “in” state requires to evaluate the limit :

$$\lim_{Z, \bar{Z} \rightarrow 0} \ln(Z \bar{Z}) \sum_{m, \bar{m}} A_{m\bar{m}}(\lambda, \bar{\lambda}) Z^{im} \bar{Z}^{i\bar{m}} \quad (4.68)$$

The powers $(Z/\bar{Z})^{im}$ are not well defined (although finite, e^{2mt}) in the $Z, \bar{Z} \rightarrow 0$ limit. the regularization prescription must be for all values of m :

$$\lim_{Z, \bar{Z} \rightarrow 0} \ln(Z \bar{Z}) A_{m, -m}(\lambda, \bar{\lambda}) Z^{im} \bar{Z}^{-im} = A_{m, -m}^{reg}(\lambda, \bar{\lambda}) e^{2mt}. \quad (4.69)$$

The presence of t in (4.69) is due to the ambiguity of the zero limit and is important.

Therefore, choosing the infinite number of “coefficients” to absorb the logarithmic singularity regularizes the expectation values of the integrals . For example, after the contour integrations absorb the singularities in the OPE of the Toda exponentials, the regularized expectation value of the zero modes of the $W_2(Z, \bar{Z})$ operator/generator is :

$$< \rho_{\lambda, \bar{\lambda}} | \hat{W}_2^{N=0, \bar{N}=0} | \rho_{\lambda, \bar{\lambda}} >_{reg} = \mathcal{F}_2[A_{m, \bar{m}}^{reg}(\lambda, \bar{\lambda})] = \Delta_1. \quad (4.70)$$

and similarly for $s = 3, 4, 5, \dots, \infty$

$$< \rho_{\lambda, \bar{\lambda}} | W_s^{N=0, \bar{N}=0} | \rho_{\lambda, \bar{\lambda}} >_{reg} = \mathcal{F}_s[A_{m, \bar{m}}^{reg}(\lambda, \bar{\lambda})] = \Delta_k \quad (4.71)$$

The r.h.s of (4.70,4.71) is given by the prescription in (4.26)). These infinite number of equations are the ones obtained after the evaluation of (4.51,4.59) using the regularization prescription (4.69) for the states as well as the point-splitting for the operators involved in the definition of the W_s generators. The \mathcal{F}_s ; for $s = 2, 3, 4, \dots, \infty$ are an infinite number of known functionals of the regularized “coefficients “ $A_{m, \bar{m}}^{reg}$ that bear the sought-after $\lambda, \bar{\lambda}$ dependence encoded in the $\varphi(t)$. Having an infinite family of functions in $\lambda, \bar{\lambda}$, $\hat{\Delta}_k^\lambda, \dots k = 1, 2, \dots$, the integral equations (4.70,4.71) for $s = 2, 3, 4, \dots$, will be sufficient (in principle) to specify $A_{m, \bar{m}}^{reg}(\lambda, \bar{\lambda})$; $m = 0, 1, 2, \dots$ enabling to establish the $|\lambda, \bar{\lambda} \rangle \rightarrow |\rho_{\lambda \bar{\lambda}} \rangle$ correspondence. Therefore, these infinite number of equations, would allow us to construct the states $|\rho_{\lambda \bar{\lambda}} \rangle$.

4.4 On W_N and A_{N-1} Casimir Algebras

If the classical Toda molecule is indeed an exact integrable system it must posses an infinite number of functionally independent classical integrals of motion whose poisson brackets are zero; i.e. in involution. At the Quantum level one should have an infinite

numbers of mutually commuting operator charges obtained from the classical quantum correspondence :

$$I_n \rightarrow Q_n; \{I_n, I_m\} = 0 \rightarrow \frac{1}{i\hbar}[Q_n, Q_m] = 0.$$

$$< \rho | Q_n | \rho > = < \rho(in) | Q_n(r \rightarrow -\infty) | \rho(in) > = < \rho(out) | Q_n(r \rightarrow +\infty) | \rho(out) > . \quad (4.72)$$

The last equation is just the expression of quantum charge conservation. The quantum integrals of motion are conserved so their expectation values in the state $|\rho\rangle$ do not depend on time, on r . An explicit example of this is the expression for the classical energy in eqs-(3.9,4.15). Hence their expectation values can be computed in terms of the asymptotic states; i.e. in terms of the function $\varphi((\lambda + \lambda^*), t)$ or its spectral decomposition $\varphi_n(\lambda + \lambda^*)$.

The Casimirs in the classical case are [4] :

$$I_n = \int_0^{2\pi} dt \left(\int dt' \varphi_{\lambda, \bar{\lambda}}(t') \right)^n. \quad (4.73)$$

The explicit expression relating the infinite number of involutive conserved *quantum* charges in terms of the generators of the chiral W_∞ algebra has been given by Wu and Yu [46] :

$$\hat{Q}_2 = \oint \hat{W}_2 dz; \hat{Q}_3 = \oint \hat{W}_3 dz. \hat{Q}_4 = \oint (\hat{W}_4 - \hat{W}_2 \cdot \hat{W}_2)(z) dz.$$

$$\hat{Q}_5 = \oint (\hat{W}_5 - 6\hat{W}_2 \cdot \hat{W}_3)(z) dz; \hat{Q}_6 = \oint (\hat{W}_6 - 12\hat{W}_2 \cdot \hat{W}_4 - 12\hat{W}_3 \cdot \hat{W}_3 + 8\hat{W}_2 \cdot \hat{W}_2 \cdot \hat{W}_2)(z) dz; \dots \quad (4.74)$$

Similar expressions hold for the antichiral algebra. In the dimensionally-reduced U_∞ algebra case, these expressions hold as well, where now the integrals to use are those of the type outlined in eqs-(4.24) which replace the contour integrals in the complex plane z, \bar{z} by integrals w.r.t. the t variable. Upon evaluation of expectation values (in the $r = -\infty$ limit for convenience) and a regularization one will have then expressions of the type :

$$E = < \hat{I}_2 >_{reg} = < \int dt \hat{W}_2[\rho(-\infty, t)] >_{reg}; < \hat{I}_3 >_{reg} = < \int dt \hat{W}_3[\rho(-\infty, t)] >_{reg} .$$

$$< \hat{I}_4 >_{reg} = < \int dt (\hat{W}_4 - \hat{W}_2 \cdot \hat{W}_2) >_{reg} . < \hat{I}_5 >_{reg} = < \int dt (\hat{W}_5 - 6\hat{W}_2 \cdot \hat{W}_3) >_{reg}; \dots \quad (4.75)$$

Recently, the construction of W_N algebras in the form of A_{N-1} Casimir algebras has been provided by Ozer and Karadayi [55]. Casimir W algebras are shown to exist in a way such that the conformal spins of the primary generating fields coincide with the orders of independent Casimirs. Ozer has shown that one can extend this relation further and construct generating fields that have *precisely* the same eigenvalues as the Casimir operators themselves. The construction can be generalized to the W_∞ case by induction.

This required a suitable change of basis [55] from the one used by Awata et al. The Casimir eigenvalues (at a given order) will then be linear combinations of the highest weights, or eigenvalues of the zero modes of the generating primary fields , Δ_k , up to that given order. And viceversa, the weights, Δ_k , can be expressed as linear combinations of the Casimir eigenvalues (up to a given order) of A_{N-1} Lie algebras in the $N \rightarrow \infty$ limit. This implies that the expectation values of the quantum conserved charges will coincide with the Casimir eigenvalues. The states $|\rho\rangle$ (in the new basis) will then be parametrized by the infinite number of Casimir eigenvalues. This is the analog of the Hydrogen atom quantum states parametrized by the quantum numbers $|n, l, m_l\rangle$. There may be other invariants besides the Casimirs (polynomials). As far as we know these have not been studied.

To summarize : given a quasi-finite highest-weight irreducible representation of the W_∞, \bar{W}_∞ algebras; i.e. given the generating function for the infinite number of conformal chiral (antichiral) weights : $\tilde{\Delta}^\lambda(x) \Rightarrow \Delta_k^\lambda$ and the central charge ; C and $b(w), \chi(characters)...$ one can (in principle) from eqs-(4.44,4.70,4.71) determine $\varphi(t)$ as a family of functions parametrized by $\lambda, \bar{\lambda}$ ($\lambda + \lambda^*$). Since the latter are continuous parameters the energy spectrum (3.9,4.15) is *continuous* in general. One has a continuum of highest weight states. In the next section we will study a simple case when one has a discrete spectrum characterized by the positive integers $n \geq 0$.

The only assumption was that the asymptotic states should be parametrized solely by the functions $\varphi(t)$. Once a particular $\varphi_{\lambda\bar{\lambda}}(t)$ is selected and related to a given $|\lambda, \bar{\lambda}\rangle$ representation, by the use of eqs-(4.44,4.70,4.71), the regularized Hamiltonian operator obeys the equation :

$$\hat{H} |\varphi_{\lambda\bar{\lambda}}\rangle = E[\varphi_{\lambda\bar{\lambda}}(t)] |\varphi_{\lambda\bar{\lambda}}\rangle . \quad (4.76)$$

. where the energy eigenvalue is given by the expectation value in eq-(4.75). An explicit and complete knowledge of $\varphi(t)$ can *only* be given once *all* the eqs-(4.44,4.70,4.71) are solved. This is not an easy task.

Instead of having quantum numbers say $|n, l, m_l\rangle$ like in the hydrogen atom, for example, here one has for “eigenvalues” integrals of suitable functions of t which are the regularized expectation values of expressions involving the operator $\hat{\rho}(r, t)$. To be able to determine $\varphi(t)$ requires solving the values of the infinite number of polynomials, A_n (after constraining $B_n = 0$) in (4.43,4.44) that play the role of the “spectra”. We proceed in **V** to find particular exact solutions in a semiclassical WKB-type approximation.

V. Discrete Spectrum

Below we will study a simple case, semiclassical WKB-type approximation, where when one has a discrete spectrum characterized by the positive integers $n \geq 0$; i.e. the $|\lambda\rangle = |n\rangle$. We shall restore now the coupling $\beta^2 < 0$ given in (2.28) . A simple fact which allows for the possibility of discrete energy states is to use the analogy of the Bohr-Sommerfield quantization condition for periodic system. It occurs if one opts to choose for the quantity $\exp[\beta\varphi(t)r] \equiv \exp[i\Omega r]$ which appears in (3.1); Ω is the frequency parameter (a constant). For this new choice of $\varphi(t)$ the expression (3.9) for the classical energy needs to be modified in general. Below we will show that if $d(t)$ is chosen to be zero then (3.9) is still valid. When the dynamical system is periodic in the variable r with periods

$2\pi/\Omega$, a way to quantize the values of Ω in units of n is to recur to the Bohr-Sommerfield quantization condition for a periodic orbit :

$$J = \oint pdq = n\hbar \quad (5.1)$$

In general the corrected Bohr-Sommerfield formula involves the inclusion of Maslov indices but we shall not be concerned about this at the moment. Including the Maslov index allowed Bohr to obtain the correct Balmer formula for the Hydrogen atom spectra.

Eq-(5.1) reflects the fact that upon emission of a quanta of energy $\hbar\Omega$ the change in the energy level as a function of n is [45] :

$$\partial E/\partial n = \hbar\Omega = \frac{2}{3}(2\pi)^3(\hbar)^2\Omega\partial\Omega/\partial n. \quad (5.2a)$$

Using (3.9) (below we will prove that eq-(3.9) is still *valid*) yields :

$$\Omega(n) = \frac{3}{2(2\pi)^3} \frac{n}{\hbar}. \quad (5.2b)$$

Hence the energy is

$$E = \frac{3}{4}(2\pi)^{-3}n^2. \quad (5.3)$$

which is reminiscent of the rotational energy levels $E \sim h^2l(l+1)$ of a rotor in terms of the angular momentum quantum numbers $l = 0, 1, \dots$. A quadratic value of $E(n) \sim n^2$ is to be expected. From eq-(4.10) one can see that Δ_1 is a quadratic polynomial in λ . In order to have a proper match of dimensions it is required to insert the membrane tension as it happens with the string. In order to classify the physical set of states we have to have at our disposal all of the unitary highest weight irrerepresentations. As far as we know these have not been constructed for W_∞ . For $W_{1+\infty}$ these have been constructed by Kac and Radul [35] and by the group [30].

These representations turned out to have a crucial importance in the classification of some of the Quantum Hall-Fluid states [39]. Saveliev [4] chose the $\varphi(t)$ in (3.1) to be negative real functions to assure that the potential term in the Hamiltonian vanished at $r \rightarrow \infty$ and arrived at (3.9). In case that the functions $\varphi(t)$ are no longer < 0 ; i.e when $\beta\varphi r$ is no longer a real valued quantity < 0 , the asymptotic formula (3.9) will no longer hold and one will be forced to perform the very complicated integral !

$$\mathcal{H} = \int dt \left[-\frac{1}{2}\beta^2(\partial p/\partial t)^2 + \left(\frac{m^2}{\beta^2}\right) \exp [\beta\partial^2 x/\partial t^2] \right]. \quad (5.4)$$

where $p = \beta\partial x/\partial r$ is the generalized momentum corresponding to $\rho \equiv \beta\partial^2 x/\partial t^2$, and $\mu^2 \equiv (\frac{m^2}{\beta^2})$ is the perturbation theory expansion parameter discussed in [33]. Without loss of generality it can be set to one. Nevertheless, eqs-(5.1,5.2) are still valid. One just needs to evaluate the Hamiltonian at $\Omega r = 2\pi p$ where p is a very large integer $p \rightarrow \infty$ and take $d(t) = 0$ in (3.2,3.3) (the logarithm is ill-defined and so is eq-(3.8b), nevertheless the energy is still finite) :

$$\exp[\partial^2 x / \partial t^2] \rightarrow d(t) \exp[i2\pi p] = 0. (\partial p / \partial t)^2 \rightarrow (\int \varphi dt')^2 \dots \quad (5.5)$$

recovering (3.9) once again.

Are there zero energy solutions ? . If one naively set $\varphi(t) \equiv 0$ in (3.2) or set $n = 0$ in (5.2) one would get a zero classical energy. However eqs-(3.2,3.3) for the most part will be singular and this would be unacceptable. One way zero energy states could be obtained is by choosing $\varphi(t), d(t)$ appropriately so that (5.4) is zero. Since one has one equation and two functions to vary presumably there should be an infinite number of solutions of zero energy. In the quantum case one has a constraint between the now operators , $\varphi(t), d(t)$, given by eq-(3.8b) through the extra c number function $w(t)$. It might be interesting to see if it is possible to set φ to zero and use $d(t), w(t)$ appropriately to avoid singularities. One still has the main problem of obtaining the operator form of the continuous Toda equation (and Toda molecule). One has to check that the operator form of these equations remains unaltered after constraining $\varphi(t)$ to zero, or for that matter, setting $d(t) = 0$.

At first sight there could be an infinity of quantum ground states. In the discrete spectrum case the lowest of the ground states (the lowest in energy of the highest weight states) corresponds to $n = 0$. From this state one then builds a representation by erecting the tower of states (4.5). As mentioned earlier we do not know whether the discrete states are the physical ones. The no-ghost theorem has yet to be constructed. For this reason it is of paramount importance to have a list of all unitary highest weight irreps in order to avoid negative norm states. In the string picture one has that the central charge must be 26 and the Regge intercept must be $a = 1$. Therefore, we expect to have a selected value for the infinite number of conformal weights Δ_k as well as value the central charge.

Solving (4.76) is analogous to solving a time independent ‘‘Schroedinger’’-like equation. Concentrating on the case that $\varphi(t) < 0$; the wave functional is defined : $\Psi[\rho, t] \equiv \langle \rho | \Psi \rangle$ where the state $|\rho \rangle_\varphi$ has an explicit dependence on φ which also depends on λ as shown in section IV. Upon replacing $\partial p / \partial t \rightarrow -i\hbar(\partial / \partial t \delta / \delta \rho)$ as an operator acting on the Ψ , the time-independent equation for the wave functional becomes :

$$\int_0^{2\pi} dt [(-i\hbar \partial / \partial t \delta / \delta \rho)^2 + \exp \rho] \Psi[\rho(r', t'); t] = \int_0^{2\pi} dt \left(\int_0^t dt' \varphi(t') \right)^2 \Psi[\rho(r', t'); t]. \quad (5.6)$$

where in this semiclassical limit one is using eqs-(3.9,4.15) for the energy. One could also have written (5.6) in the momentum representation : $\rho \rightarrow -i\hbar \delta / \delta p$ acting on the ‘‘Fourier’’ transform of Ψ .

The action functional is :

$$\int dt \int \mathcal{D}\rho dr \Psi^+ (i\hbar \frac{\partial}{\partial r} - \mathcal{H}) \Psi[\rho(r', t'); r, t]. \quad (5.7)$$

$\mathcal{D}\rho$ is the functional integration measure; r is the variable linked with the physical time and the on-shell condition is just (5.6). This is the second-quantization of the physical quantities. One must *not* interpret Ψ as a probability amplitude but as a field operator which creates a continuous Toda field in a given quantum state $|\rho \rangle_\varphi$ associated with the

classical configuration configuration given by eq-(3.1) in terms of $\varphi_\lambda(t)$. The functional differential equation (5.6) is extremely complicated. A naive zeroth-order simplification will be given shortly. This is because the Ψ can have the form $\Psi = \Psi[\rho, \rho_{t'}, \rho_{t't'}, \dots]$. The equation in the momentum representation does not have that complexity but it has an exponential functional differential operator. Even further, the Ψ is a non-local object. In string field theory the string field is a multilocal object that depends on all of the infinite points along the string.

One can expand $|\Psi\rangle$ in an infinite dimensional basis spanned by the Verma module (4.5) associated with the state $|\lambda\rangle$ and let λ run as well over all the highest weights. Given a vector $v_\lambda \in \mathcal{V}_\lambda$ (Verma module) one has :

$$|\Psi\rangle = \sum_{\lambda} \sum_{v_\lambda} \langle v_\lambda | \Psi \rangle |v_\lambda\rangle \cdot v_\lambda \in \mathcal{V}_\lambda \quad (5.8)$$

This is very reminiscent of the string-field $\Phi[X(\sigma)] = \langle X | \Phi(x_o) \rangle$ where x_o is the center of mass coordinate of the string and the state $|\Phi(x_o)\rangle$ is comprised of an infinite array of point fields :

$$|\Phi(x_o)\rangle = \phi(x_o)|0\rangle + A_\mu(x_o)a_1^{\mu+}|0\rangle + g_{\mu\nu}a_1^{\mu+}a_1^{\nu+}|0\rangle + \dots \quad (5.9)$$

where the first field is the tachyon, the second is the massless Maxwell, the third is the massive graviton....In the string case one does not customary expand over the towers of the Verma module since these states have zero norm. However one should include all of the states. The oscillators play the role of ladder-like operators acting on the "vacuum" $|0\rangle$ in the same manner that the Verma module is generated from the highest weight state $|\lambda\rangle$ by successive application of a string of $W(z^{-n}D^k)$ operators acting on $|\lambda\rangle$. The state $|\rho(r, t)\rangle_\varphi$ It is the relative of the string state $|X(\sigma_1, \sigma_2)\rangle$ whereas $|\Psi\rangle$ is the relative of the string field state $|\Phi\rangle$. The naive zeroth order approximation of the "Schroedinger"-like equation is of the form :

$$[\partial_t^2 \partial_y^2 + e^y] \Psi(y, t) = E \Psi(y, t). \quad (5.10)$$

A change of variables $:x = 2e^{y/2}$ converts (5.10) into Bessel's equation after one sets $\Psi(y, t) = e^t \Phi(y)$ or equal to $e^{-t} \Phi(y)$:

$$(x^2 \partial_x^2 + x \partial_x + x^2 - 4E) \Phi(x) = 0. \quad (5.11)$$

and whose solution is $\Phi(x) = c_1 \mathcal{J}_\nu(x) + c_2 \mathcal{J}_{-\nu}(x)$ where $\nu \equiv 2\sqrt{E}$ and c_1, c_2 constants. The wavefunctional is then a linear combination of :

$$\Psi[\rho(r', t'); t] = e^t \int \int dr' dt' [c_1 \mathcal{J}_\nu(2e^{\rho(r', t')/2}) + c_2 \mathcal{J}_{-\nu}(2e^{\rho(r', t')/2})] \quad (5.12)$$

or the other solution involving e^{-t}

One may notice that discrete energy level (in suitable units such as $\nu = 2\sqrt{E} = n$) solutions are possible. Earlier we saw in (5.2) that $E(n) \sim n^2$ so $\sqrt{E} \sim n$. Therefore

setting $2\alpha e^{y/2\alpha} = x$ where α is a suitable constant allows to readjust $\nu = 2\sqrt{\alpha E} = n$. The Bessel functions will have nodes at very specific points. The solutions in this case will be given in terms of \mathcal{J}_n and the modified Bessel function of the second kind, K_n . These solutions are tightly connected with the boundary conditions of the wave-functional. This concludes the study of this simple model.

VI. CONCLUSION AND CONCLUDING REMARKS

An exact integrable set of quantized solutions of the spherical membrane moving in flat target backgrounds has been studied. Such a special class of solutions are those related to dimensional reductions of $SU(\infty)$ YM theories from ten to four dimensions. The latter are constructed in terms of instanton solutions in $D = 4$ that can be related to the continuous Toda theory after an ansatz is made. Not surprisingly, the system is integrable. We don't know the fate of more general class of solutions of the YM equations nor what happens for surfaces whose topology is not spherical nor the case of membranes moving in curved backgrounds. This deserves further study. We hope to have advanced the need to built unitary highest weights irreps of W_∞ algebras

Membranes appear in a wide variety of physical models .

1- As boundaries of a four dimensional anti-deSitter spacetime, AdS_4 , [48].

2- as a coherent state of an infinite number of strings . This is reminiscent of the Sine-Gordon soliton being the fundamental fermion of the massive Thirring model, a quantum lump [45]. The lowest fermion-antifermion bound state (soliton-antisoliton doublet) is the fundamental meson of Sine-Gordon theory. Higher level states are built from excitations of the former in the same way that infinitely many massless states can be built from just two singletons.

3- As a Matrix [53] model. Uncompactified $D = 11$ M -theory was found to have an equivalence with the $N = \infty$ limit of supersymmetric matrix quantum mechanics describing $D = 0$ branes. Matrix models of $2D$ gravity and Toda theory have been discussed by Gerasimov et al [57] and by Kharchev et al [51].

3-In Edge states of Quantum Hall Fluids[39] : the set of unitary highest weight irreps of $W_{1+\infty}$ have been used to algebraically characterize the low energy edge-excitations of the incompressible (area preserving) Quantum Hall Fluids.

4-Perhaps the most relevant physical applications of the membrane quantization program will be in the behaviour of black hole horizons [36]. The connection between black hole physics and non-abelian Toda theory has been studied in [37]. W gravity was formulated as chiral embeddings of a Riemann surface into CP^n . Toda theory plays a crucial role as well [38].

5-. In understanding the string vacua : The ordinary bosonic string has been found to be a special vacua of the $N = 1$ superstring [40]. It appears that there is a whole hierarchy of string theories : w_2 string is a particular vacua of the w_3 string and so forth.....If this is indeed correct one has then that the (super) membrane, viewed as noncritical W_∞ string theory, is, in this sense, the universal space of string theory. The fact, advocated by many, that a Higgs symmetry-breakdown-mechanism of the infinite number of massless states of the membrane generates the infinite massive string spectrum fits within this description.

6- As gauge theories of area preserving diffs [41] and as composite antisymmetric tensor field theories [42,43].

And many more. We hope that the essential role that Self Dual $SU(\infty)$ Yang-Mills theory has played in the origins of the membrane-Toda theory, will shed more light into the origin of duality in string theory and the full membrane spectrum. In [43] the analogs of S, T duality were built in from the very start. For a review of duality in string theory [50] and the status of string solitons see [49]. An important review of extended conformal field theories see [28].

As of now we must have all unitary irreps of W_∞ to construct no-ghost theorems and be able to have the OPE of the Toda exponentials to fully exploit the results of **IV**. The discrete spectrum solution warrants a further investigation and the supersymmetric sector as well.

ACKNOWLEDGEMENTS. We thank M.V. Saveliev for many helpful suggestions concerning the exact quantization program of the continuous Toda theory. To L. Boya for explaining the Maslov index to me in connection to the WKB approximation. To G.Sudarshan, Y.Ne'eman, C.Ordonez, J.Pecina, B. Murray for discussions and to the Towne family for their kind and warm hospitality in Austin, Texas. This work was supported in part by a ICSC, World Laboratory Fellowship.

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